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Subdirectly irreducible semilattices with an automorphism

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Let us denote by **SA** the variety of universal algebras with one binary operation $\wedge$ and two unary operations f, f^{-1} satisfying the equations

$$\begin{aligned}
&(x \wedge y) \wedge z = x \wedge (y \wedge z),\\
&x \wedge y = y \wedge x,\\
&x \wedge x = x,\\
&f(x \wedge y) = f(x) \wedge f(y),\\
&f^{-1}(f(x)) = f(f^{-1}(x)) = x.
\end{aligned}$$

In other words, **SA** is the variety of semilattices with one automorphism (which is, as well as its inverse, considered as an additional fundamental operation). The aim of this paper is to find all subdirectly irreducible algebras in **SA**.

A universal algebra $\mathcal{A}$ is said to be subdirectly irreducible (shortly, an SI algebra) if it contains more than one element and among all the nontrivial congruences of $\mathcal{A}$ there exists a least one; nontrivial means different from $id_{\mathcal{A}} = \{(a,a); a \in \mathcal{A}\}$.

The largest example of an SI algebra in **SA** is the algebra $\mathcal{P}(Z)$ defined as follows. Its underlying set is the set of all subsets of Z (where Z denotes the set of integers); the operations are defined by

$$\begin{aligned}
&A \wedge B = A \cap B,\\
&f(A) = A + 1 = \{a+1; a \in A\},\\
&f^{-1}(A) = A - 1 = \{a-1; a \in A\}.
\end{aligned}$$

The least nontrivial congruence of $\mathcal{P}(Z)$ is the congruence α defined by $(A,B) \in \alpha$ iff either $A = B$ or both A and B are at most one-element subsets of Z.

We shall prove in Section 1 that every SI algebra from **SA** can be embedded into $\mathcal{P}(Z)$. This means that the variety **SA** is residually small, i.e.,

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there is an upper bound for the cardinalities of its SI members. As it is proved in W. Taylor [3], in the case of a finite similarity type a residually small variety cannot contain SI algebras of any cardinality greater than $2^{\aleph_0}$. Our variety **SA** belongs thus to the larger residually small varieties, as it contains uncountable SI algebras.

There are uncountably many SI algebras in **SA** and so it is not possible to give their list. When we say that we shall find all of them, we mean the following. First of all, we shall show that every such algebra can be embedded into $\mathcal{P}(Z)$. Then it will turn out that the collection of the SI subalgebras of $\mathcal{P}(Z)$ can be decomposed into the union of a countable family of pairwise disjoint intervals in the subalgebra lattice of $\mathcal{P}(Z)$; and we shall be able to give a list of all these intervals. Further we shall show that two distinct SI subalgebras of $\mathcal{P}(Z)$ can never be isomorphic.

This paper is related to the paper [1] in which all simple semilattices with two commuting automorphisms are found, and also to the paper [2] in which Boolean algebras are considered.

1. Embedding into $\mathcal{P}(Z)$

Proposition 1.1. *Let $\mathcal{A}$ be an algebra from* **SA** *and a be any element of $\mathcal{A}$. Define a mapping h of $\mathcal{A}$ into $\mathcal{P}(Z)$ by $h(x) = \{i \in Z; f^i(a) \leq x\}$ for any $x \in \mathcal{A}$. Then h is a homomorphism of $\mathcal{A}$ into $\mathcal{P}(Z)$. Moreover, if there exists an element $b \in \mathcal{A}$ such that $a \not\leq b$ and the pair (a, b) belongs to any nontrivial congruence of $\mathcal{A}$ then h is an isomorphism onto a subalgebra of $\mathcal{P}(Z)$.*

Proof. We have

$$\begin{aligned} h(x \wedge y) =& \{i; f^i(a) \leq x \wedge y\} = \{i; f^i(a) \leq x\} \cap \{i; f^i(a) \leq y\} = \\ & h(x) \cap h(y), \\ h(f(x)) =& \{i; f^i(a) \leq f(x)\} = \{i; f^{i-1}(a) \leq x\} = \\ & \{i+1; i \in h(x)\} = h(x) + 1 = f(h(x)) \end{aligned}$$

and similarly $h(f^{-1}(x)) = f^{-1}(h(x))$. This proves that h is a homomorphism, so that the kernel of h is a congruence. Now, if $a \not\leq b$ then (a, b) cannot belong to this kernel, since $0 \in h(a)$ and $0 \notin h(b)$. So, if (a, b) belongs to any nontrivial congruence of $\mathcal{A}$ then the kernel of h is trivial, which means that h is injective.∎

Proposition 1.2. *Let $\mathcal{A}$ be an SI algebra from* **SA**. *Then $\mathcal{A}$ is isomorphic to a subalgebra of $\mathcal{P}(Z)$.*

Proof. The least nontrivial congruence of $\mathcal{A}$ contains a pair (a, b) such that $a \not\leq b$. It remains to apply 1.1. ∎

2. Necessary conditions for a subalgebra of $\mathcal{P}(Z)$ to be SI

In this section let $\mathcal{A}$ be an SI subalgebra of $\mathcal{P}(Z)$. We shall denote the least nontrivial congruence of $\mathcal{A}$ by α. The algebra $\mathcal{A}$ may but need not contain the smallest element; if it exists, the smallest element is the empty set.

By a generic element (of $\mathcal{A}$) we shall mean an element $M \in \mathcal{A}$ for which there exists another element $M' \in \mathcal{A}$ such that M' is a proper subset of M and $(M', M) \in \alpha$. It is evident that $\mathcal{A}$ contains at least one generic element and that any generic element is a nonempty set.

Lemma 2.1. *Let M be a generic element. The following are true for any two elements $A, B \in \mathcal{A}$:*

(1) $A = B$ iff $M + i \subseteq A$ is equivalent to $M + i \subseteq B$ for any $i \in Z$.
(2) $A \subseteq B$ iff $M + i \subseteq A$ implies $M + i \subseteq B$ for any $i \in Z$.
(3) $A \neq \emptyset$ implies $M + i \subseteq A$ for some $i \in Z$.

Proof. (1) follows from 1.1 and (2),(3) are consequences of (1). ∎

The period of a generic element M is the nonnegative integer k defined as follows. If M is a one-element set, then $k = 0$; otherwise, k is the least positive integer for which there exist two elements $a, b \in M$ with $|a - b| = k$. For $k > 0$ put $C_k = \{i \in Z; i \equiv 0 \bmod k\}$.

Lemma 2.2. *Let M_0 and M be two generic elements; denote by k the period of M_0. Then there exists a nonnegative integer p such that one of the following two cases takes place:*

(1) $a + qk \in M$ for any $a \in M$ and any integer $q \geq p$;
(2) $a - qk \in M$ for any $a \in M$ and any integer $q \geq p$.

Proof. Since M is not the least element of $\mathcal{A}$, it follows from 2.1(3) that there exists an integer i with $M_0 + i \subseteq M$. Now, M_0 contains two elements a, b with $|a - b| = k$; using $M_0 + i \subseteq M$ we get that the same must be true for the set M. This means that the intersection $M \cap (M - k)$ is nonempty. One can easily see that no other set than $\emptyset$ can be the least element of $\mathcal{A}$. So, $M \cap (M - k)$ is not the least element and now it follows from 2.1(3) that there exists an integer j with $M + j \subseteq M \cap (M - k)$. In other words, there exists a $j \in Z$ such that whenever $a \in M$ then $a + j \in M$ and $a + j + k \in M$. Evidently, every multiple of j by a positive integer has the same property and so we can assume that j is a multiple of k. If $j = 0$ then (1) is true with $p = 0$. Let $j \neq 0$ (so that $k \neq 0$ too). For $a \in M$ we have

$$a + j \in M, \quad a + j + k \in M,$$
$$a + 2j \in M, \quad a + 2j + k \in M, \quad a + 2j + 2k \in M,$$
$$a + 3j \in M, \quad a + 3j + k \in M, \quad a + 3j + 2k \in M, \quad a + 3j + 3k \in M,$$

etc. From this we see that if $j > 0$ then (1) is true with $p = j^2/k^2$ and if $j < 0$ then (2) is true with, again, $p = j^2/k^2$. ∎

Lemma 2.3. *All generic elements of $\mathcal{A}$ have the same period k. Moreover, exactly one of the following four cases takes place:*

(1) every generic element is a one-element set;

(2) $k > 0$ and every generic element equals $C_k + i$ for a number $i \in \{0, \dots, k-1\}$;

(3) $k > 0$ and every generic element M is a lower bounded set and contains an element a such that all the elements $a, a+k, a+2k, \dots$ belong to M;

(4) $k > 0$ and every generic element M is an upper bounded set and contains an element a such that all the elements $a, a-k, a-2k, \dots$ belong to M.

Proof. It follows from 2.2. ∎

Lemma 2.4. *Let M be a generic element with period $k > 0$. Then $a \equiv b \pmod k$ for any two elements $a, b \in M$.*

Proof. It is sufficient to consider the case (3) in Lemma 2.3 only. Denote by c the least element of M such that all the elements $c, c+k, c+2k, \dots$ belong to M. Let $a \in M$ be arbitrary. By 2.2 there exists a positive integer q such that $a + qk \in M$ and $a + qk \geq c$. We have $c + pk \leq a + qk < c + pk + k$ for a nonnegative integer p. Since both $c + pk$ and $a + qk$ belong to M, it follows from the definition of k that $c + pk = a + qk$. We get $a \equiv c \pmod k$ for any $A \in M$. ∎

Lemma 2.5. *Let M be a generic element with period $k > 0$ and let $A \in \mathcal{A}$ be a set containing a least element a; let $e \equiv a \pmod k$ for all $e \in A$. Then the set $A \setminus \{a\}$ belongs to $\mathcal{A}$.*

Proof. Case (3) of Lemma 2.3 is the only possible here. Denote by p the unique integer such that $p \notin M$ and all the elements $p+k, p+2k, p+3k, \dots$ belong to M. We have $A \setminus \{a\} = A \cap (M + a - p)$. ∎

Lemma 2.6. *Let M, M' be two generic elements of $\mathcal{A}$. Then $M' = M + i$ for an integer i.*

Proof. It is sufficient to consider the case (3) in Lemma 2.3 and to prove that if M, M' are two generic elements such that 0 is the least element of both M and M' then $M \subseteq M'$. By 2.5 we have $M' \setminus \{0\} \in \mathcal{A}$. For $i > 0$ the set $M + i$ does not contain 0 and so $M + i \subseteq M'$ is true iff $M + i \subseteq M' \setminus \{0\}$. By 2.1(1) there exists an $i \leq 0$ such that $M + i \subseteq M'$. Clearly, only $i = 0$ is possible. But then $M \subseteq M'$. ∎

Lemma 2.7. *In the case (2) of Lemma 2.3, every element of $\mathcal{A}$ is equal to $(C_k + i_1) \cup \dots \cup (C_k + i_r)$ for a subset $\{i_1, \dots, i_r\}$ of $\{0, \dots, k-1\}$.*

Proof. It suffices to prove that if $A \in \mathcal{A}$ and $a \in A$ then $C_k + a \subseteq A$. The set $A \cap (C_k + a)$ is nonempty and belongs to $\mathcal{A}$. However, it follows easily from 2.1 that no proper nonempty subset of $C_k + a$ can belong to $\mathcal{A}$. We get $A \cap (C_k + a) = C_k + a$ and thus $C_k + a \subseteq A$. ∎

Lemma 2.8. *Let the case (3) of Lemma 2.3 take place and let M_0 be the unique generic element of $\mathcal{A}$ with the least element 0. If $A \in \mathcal{A}$ and $a \in A$ then $M_0 + a \subseteq A$.*

Proof. Put $B = A \cap (M_0 + a)$, so that a is the least element of B. Since B is contained in $M_0 + a$, by 2.4 we get $e \equiv a \pmod{k}$ for all $e \in B$. Consequently, Lemma 2.5 yields $B \setminus \{a\} \in \mathcal{A}$. For $i > 0$ we have $a \notin M_0 + a + i$, so that $M_0 + a + i \subseteq B$ is true iff $M_0 + a + i \subseteq B \setminus \{a\}$. It follows, using 2.1, that there exists an $i \leq 0$ with $M_0 + a + i \subseteq B$. Clearly, $i = 0$ and we get $M_0 + a \subseteq B \subseteq A$. ∎

3. Sufficient conditions for a subalgebra of $\mathcal{P}(Z)$ to be SI

Lemma 3.1. *Let $\mathcal{A}$ be a subalgebra of $\mathcal{P}(Z)$ containing at least one one-element set (so that it contains all the one-element subsets, the empty set and perhaps some more subsets of Z). Then $\mathcal{A}$ is subdirectly irreducible; a pair $(A, B) \in \mathcal{A}^2$ belongs to the least nontrivial congruence of $\mathcal{A}$ iff either $A = B$ or both A and B are at most one-element sets.*

Proof. It is obvious. ∎

For $k \geq 1$ we denote by $\mathcal{B}_k$ the set of the subsets A of Z such that whenever $a \in A$ then $C_k + a \subseteq A$.

Lemma 3.2. *Let $k \geq 1$. Then $\mathcal{B}_k$ is a subalgebra of $\mathcal{P}(Z)$ and any subalgebra $\mathcal{A}$ of $\mathcal{B}_k$ containing both C_k and $\emptyset$ as elements is subdirectly irreducible; a pair $(A, B) \in \mathcal{A}^2$ belongs to the least nontrivial congruence of $\mathcal{A}$ iff either $A = B$ or both A and B belong to the set $\{\emptyset, C_k, C_k + 1, \ldots, C_k + k - 1\}$.*
(Notice that for $k \geq 2$, the condition $\emptyset \in \mathcal{A}$ is a consequence of $C_k \in \mathcal{A}$.)

Proof. It is obvious. ∎

Let $k \geq 1$. By a positively k-generic subset of Z we shall mean a subset $M \subseteq Z$ satisfying the following four conditions:

(1) 0 is the least element of M;
(2) $M \subset C_k$;
(3) there exists an element a of M such that $C_k \cap [a) \subseteq M$;
(4) if $a \in M$ then $M + a \subseteq M$.

Given a positively k-generic subset M of Z, we denote by $\mathcal{B}_{k,M}$ the set of the subsets A of Z such that whenever $a \in A$ then $M + a \subseteq A$.

Lemma 3.3. *$\mathcal{B}_{k,M}$ is a subalgebra of $\mathcal{P}(Z)$ and the sets $\emptyset, Z, M, C_k \cap [0)$ and $M \setminus \{0\}$ belong to $\mathcal{B}_{k,M}$.*

Proof. It is easy to see that $\mathcal{B}_{k,M}$ is a subalgebra containing $\emptyset$ and Z. The fact that M belongs to $\mathcal{B}_{k,M}$ follows from (4). Further, with $a \in M$ such that $C_k \cap [a) \subseteq M$ we have

$$C_k \cap [0) = (M \cap (M - k) \cap \ldots \cap (M - a)) - a$$

and

$$M \setminus \{0\} = M \cap ((C_k \cap [0)) + k). \qquad ∎$$

Lemma 3.4. *Let M be a positively k-generic subset of Z such that $M \neq C_k \cap [0)$ and let $\mathcal{A}$ be any subalgebra of $\mathcal{B}_{k,M}$ containing M. Define a binary relation α on $\mathcal{A}$ as follows: $(A, B) \in \alpha$ iff either $A = B$ or $\{A, B\} = \{M + i, (M \setminus \{0\}) + i\}$ for some $i \in Z$. Then α is just the least nontrivial congruence of $\mathcal{A}$.*

Proof. It is easy to see that α is an equivalence; its transitivity follows from the assumption $M \neq C_k \cap [0)$, as this implies that we can never have $M + i = (M \setminus \{0\}) + j$ for two integers i and j. Also, it is evident that if $(A, B) \in \alpha$ then $(A + 1, B + 1) \in \alpha$ and $(A - 1, B - 1) \in \alpha$. In order to finish the verification of the congruence property of α, it is sufficient to prove $(M \cap A, (M \setminus \{0\}) \cap A) \in \alpha$ for any $A \in \mathcal{A}$. If $0 \notin A$ then $M \cap A = (M \setminus \{0\}) \cap A$. So, let $0 \in A$. By the definition of $\mathcal{B}_{k,M}$ we have $M + 0 \subseteq A$, which means that $M \subseteq A$. But then $(M \cap A, (M \setminus \{0\}) \cap A) = (M, M \setminus \{0\}) \in \alpha$.

Let now β be any nontrivial congruence of $\mathcal{A}$. There exists a pair $(A, B) \in \beta$ with $A \not\subseteq B$. Take an element $a \in A$ such that $a \notin B$. We have $0 \in A - a$ and so $M \subseteq A - a$ by the definition of $\mathcal{B}_{k,M}$. Since β is a congruence, $((A - a) \cap M, (B - a) \cap M) \in \beta$; that is, $(M, (B - a) \cap M) \in \beta$. Now, $(B - a) \cap M$ is a subset of M not containing 0 and hence $(B - a) \cap M \subseteq M \setminus \{0\}$. From this we get $(M, M \setminus \{0\}) \in \beta$. But then $(M + i, (M \setminus \{0\}) + i) \in \beta$ and thus $\alpha \subseteq \beta$. ∎

Lemma 3.5. *Let $k \geq 1$ and $M = C_k \cap [0)$, so that M is a positively k-generic subset of Z. Let $\mathcal{A}$ be any subalgebra of $\mathcal{B}_{k,M}$ containing M. Define a binary relation α on $\mathcal{A}$ as follows: $(A, B) \in \alpha$ iff either $A = B$ or there exist two integers i, j such that $i \equiv j \pmod{k}$, $A = M + i$ and $B = M + j$. Then α is just the least nontrivial congruence of $\mathcal{A}$.*

Proof. Evidently, α is an equivalence and $(A, B) \in \alpha$ implies both $(A+1, B+1) \in \alpha$ and $(A-1, B-1) \in \alpha$. In order to prove that α is a congruence, we need to show that if $A \in \mathcal{A}$, $i \equiv j \pmod{k}$ and $i < j$ then $((M+i) \cap A, (M+j) \cap A) \in \alpha$. If none of the numbers $i, i+k, i+2k, \dots, j-k$ belongs to A then $(M+i) \cap A = (M + j) \cap A$. If some does belong, denote the least of them by p; by the definition of $\mathcal{B}_{k,M}$ all the numbers $p, p + k, p + 2k, \dots$ belong to A and so $((M + i) \cap A, (M + j) \cap A) = (M + p, M + j) \in \alpha$.

Quite similarly as in the proof of 3.4, if β is any nontrivial congruence of $\mathcal{A}$ then $(M, M \setminus \{0\}) \in \beta$, i.e., $(M, M + k) \in \beta$. But then it is easy to see that $(M + i, M + j) \in \beta$ whenever $i \equiv j \pmod{k}$, so that $\alpha \subseteq \beta$. ∎

Lemma 3.6. *Let M be a positively k-generic subset of Z. Then any subalgebra of $\mathcal{B}_{k,M}$ containing M is subdirectly irreducible.*

Proof. It is a consequence of 3.4 and 3.5. ∎

4. Formulation of the result

Denote by $\mathcal{I}$ the set of the ordered triples (k, r, I) such that k is a positive integer, r is a nonnegative integer, I is a finite set of nonnegative integers and either $(r, I) = (0, \emptyset)$ or the following are true:

(1) $r \geq 2$;

(2) $0 \in I \subseteq \{0, k, 2k, \dots, (r-2)k\}$;

(3) if $a \in I$ then $I + a \subseteq I \cup \{rk, (r+1)k, (r+2)k, \dots, a + (r-2)k\}$.

For any triple $(k, r, I) \in \mathcal{I}$, the union $I \cup \{rk, (r+1)k, \dots\}$ is a k-generic subset of Z in the sense of Section 3. Conversely, given a k-generic subset M, a triple $(k, r, I) \in \mathcal{I}$ can be defined in this way: r is the least nonnegative integer such that $\{rk, (r+1)k, \dots\} \subseteq M$, and $I = M \setminus \{rk, (r+1)k, \dots\}$. We get a one-to-one correspondence between elements of $\mathcal{I}$ and k-generic subsets of Z. The only reason why we formulate the next result in terms of the triples (k, r, I) instead of the k-generic subsets is that the triples are finite objects, while k-generic subsets are not.

Theorem 4.1. *Any subdirectly irreducible algebra from* **SA** *is isomorphic to exactly one subalgebra of $\mathcal{P}(Z)$. A subalgebra of $\mathcal{P}(Z)$ is subdirectly irreducible iff it belongs to one of the intervals in the subalgebra lattice of $\mathcal{P}(Z)$ listed here:*

(1) the interval $[\mathcal{U}, \mathcal{P}(Z)]$ where $\mathcal{U} = \{\emptyset\} \cup \{\{i\}; i \in Z\}$;

(2) for any $k \geq 1$, the interval $[\mathcal{A}_k, \mathcal{B}_k]$ where $\mathcal{A}_k = \{\emptyset\} \cup \{C_k, C_k + 1, \dots, C_k + k - 1\}$ and $\mathcal{B}_k = \{(C_k + i_1) \cup \ldots \cup (C_k + i_m); m \geq 0 \text{ and } i_1, \dots, i_m \in \{0, \dots, k-1\}\}$;

(3) for any $e = (k, r, I) \in \mathcal{I}$, the interval $[\mathcal{A}_e, \mathcal{B}_e]$ where $\mathcal{A}_e$ is the subalgebra of $\mathcal{P}(Z)$ generated by the element $I \cup \{rk, (r+1)k, \dots\}$ and $\mathcal{B}_e$ is the set of all the subsets A of Z such that whenever $a \in A$ then $I + a \subseteq A$ and all the numbers $a + rk, a + (r+1)k, \dots$ belong to A;

(4) for any $e = (k, r, I) \in \mathcal{I}$, the interval $[\mathcal{A}'_e, \mathcal{B}'_e]$ where $\mathcal{A}'_e$ is the subalgebra of $\mathcal{P}(Z)$ generated by $\{-i; i \in I\} \cup \{-rk, -(r+1)k, \dots\}$ and $\mathcal{B}'_e$ is the set of all the subsets A of Z such that whenever $a \in A$ then $\{-i; i \in I\} + a \subseteq A$ and all the numbers $a - rk, a - (r+1)k, \dots$ belong to A.

Moreover, all these intervals are pairwise disjoint.

Proof. The fact that a subalgebra of $\mathcal{P}(Z)$ is subdirectly irreducible iff it belongs to one of the intervals in the list is just what we proved in the Sections 2 and 3. Taking 1.2 into account, it remains to prove that if $\mathcal{A}, \mathcal{B}$ are two SI subalgebras of $\mathcal{P}(Z)$ either belonging to two different intervals or belonging to one but not being equal then $\mathcal{A}, \mathcal{B}$ are not isomorphic.

A characteristic property of subalgebras in $[\mathcal{U}, \mathcal{P}(Z)]$ is that they are infinite and contain atoms. If h is an isomorphism between two such subalgebras $\mathcal{A}$ and $\mathcal{B}$ then $h(\{0\}) = \{a\}$ for an integer a and the mapping $h' = f^{-a}h$ is an isomorphism such that $h'(\{0\}) = \{0\}$; we get $h'(\{i\}) = \{i\}$ for all $i \in Z$; but then $h'(A) = A$ for all $A \in \mathcal{A}$ and we get $\mathcal{A} = \mathcal{B}$.

Let $k \geq 1$. The subalgebras in $[\mathcal{A}_k, \mathcal{B}_k]$ are characterized by being finite and containing exactly k atoms; it is clear that they are pairwise nonisomorphic.

An algebra $\mathcal{A} \in [\mathcal{A}_e, \mathcal{B}_e]$ can never be isomorphic to an algebra $\mathcal{B} \in [\mathcal{A}'_d, \mathcal{B}'_d]$, since it does not share with $\mathcal{B}$ the following property: there exists a positive integer i such that $f^i(A) < A$ for all nonextreme elements $A \in \mathcal{A}$.

Let $e_1 = (k_1, r_1, I_1)$ and $e_2 = (k_2, r_2, I_2)$ be two triples from $\mathcal{I}$ and let h be an isomorphism of an algebra $\mathcal{A} \in [\mathcal{A}_{e_1}, \mathcal{B}_{e_1}]$ onto an algebra $\mathcal{B} \in [\mathcal{A}_{e_2}, \mathcal{B}_{e_2}]$; we must prove that then $e_1 = e_2$ and $\mathcal{A} = \mathcal{B}$. For $i = 1, 2$ put $M_i = I_i \cup \{r_i k_i, (r_i + 1)k_i, \dots\}$. Since M_1 is an element of $\mathcal{A}$ standing with a smaller element in the least nontrivial congruence and since the only elements of $\mathcal{B}$ with the same property are the sets $M_2 + j$ $(j \in Z)$, there exists an integer j such that $h(M_1) = M_2 + j$. Now, the mapping $h' = f^{-j}h$ is an isomorphism of $\mathcal{A}$ onto $\mathcal{B}$ such that $h'(M_1) = M_2$. Let $A \in \mathcal{A}$. For any integer i we have $M_1 + i \subseteq A$ iff $h'(M_1) + i \subseteq h'(A)$, i.e., iff $M_2 + i \subseteq h'(A)$. But $M_1 + i \subseteq A$ is equivalent to $i \in A$ and $M_2 + i \subseteq h'(A)$ is equivalent to $i \in h'(A)$. We get $A = h'(A)$. This proves $\mathcal{A} = \mathcal{B}$; we also get $M_1 = M_2$ and hence $e_1 = e_2$. ∎

Proposition 4.2. *The algebras in the interval $[\mathcal{U}, \mathcal{P}(Z)]$ are all infinite and there are both countable ones and uncountable ones among them. The algebras in $[\mathcal{A}_k, \mathcal{B}_k]$ for $k \geq 1$ are all finite. The algebras in $[\mathcal{A}_e, \mathcal{B}_e]$ (and in $[\mathcal{A}'_e, \mathcal{B}'_e]$, as well) are all countably infinite for any $e \in \mathcal{I}$.*

Proof. As the rest is obvious, we need only to prove that if $e = (k, r, I) \in \mathcal{I}$ then the algebra $\mathcal{B}_e$ is countable. But this follows from the fact that any set $A \in \mathcal{B}_e$ is uniquely determined by its intersections with the sets $C_k, C_k + 1, \dots, C_k + k - 1$ and any of these intersections $A \cap (C_k + i)$ is uniquely determined by the ordered pair (a, X) where a is the least element of A (or either $-\infty$ or $+\infty$ in the cases $A \cap (C_k + i) = C_k + i$ and $A \cap (C_k + i) = \emptyset$) and $X = A \cap \{a, a + k, \dots, a + rk\}$ (or $X = \emptyset$ if $a \notin Z$). ∎

There are some open questions left:

Are the intervals $[\mathcal{A}_e, \mathcal{B}_e]$ countable for any $e \in \mathcal{I}$?

Is it possible to find all subdirectly irreducible semilattices with an endomorphism, and all subdirectly irreducible semilattices with two commuting automorphisms?

References

1 Ježek, J., *Simple semilattices with two commuting automorphisms*, Algebra Universalis **15** (1982), 162–175.

2 Ježek, J., *Subdirectly irreducible and simple Boolean algebras with endomorphisms*, Proceedings, Charleston 1984, Universal Algebra and Lattice Theory, 150–162. Lecture Notes in Math. **1149**, Springer-Verlag, 150–162.

3 Taylor, W., *Residually small varieties*, Algebra Universalis **2** (1972), 33–53.

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