THE VARIETY GENERATED BY EQUIVALENCE ALGEBRAS

J. Ježek and R. McKenzie

Abstract. Every equivalence relation can be made into a groupoid with the same underlying set if we define the multiplication as follows: $xy = x$ if x, y are related; otherwise, $xy = y$. The groupoids, obtained in this way, are called equivalence algebras. We find a finite base for the equations of equivalence algebras. The base consists of equations in four variables, and we prove that there is no base consisting of equations in three variables only. We also prove that all subdirectly irreducibles in the variety generated by equivalence algebras are embeddable into the three-element equivalence algebra, corresponding to the equivalence with two blocks on three elements.

0. Introduction

By a digraph we mean a directed graph with loops, i.e., a set equipped with a reflexive binary relation $\rightarrow$. By a quasitrivial groupoid we mean a groupoid G such that $xy \in \{x, y\}$ for all $x, y \in G$. (A groupoid is an algebra with one binary operation, which can be denoted multiplicatively.) There is a one-to-one correspondence between digraphs and quasitrivial groupoids: If $(D, \rightarrow)$ is a digraph, we define multiplication on D by $xy = x$ whenever $x \rightarrow y$, and $xy = y$ otherwise; on the other hand, every quasitrivial groupoid can be made into a digraph by setting $x \rightarrow y$ if and only if $xy = x$. In this way we can identify digraphs with quasitrivial groupoids. Quasitrivial groupoids will be also called digraph algebras.

In a number of papers, for example, [2], [3], [4], [5], [6], [8], [9], [12] and [13], commutative digraphs are investigated from the algebraic point of view. (Commutative digraphs are called tournaments). The other interesting classes of digraphs (like posets, quasi-orderings, tolerances or equivalences) have been so far much neglected in this respect; the only two related papers seem to be [7] and [10].

It has been proved in [8] that the variety of groupoids generated by digraphs, as well as the variety generated by tournaments, are not finitely based. The question is natural to ask for other interesting classes of digraphs. In this paper we will prove that the variety generated by equivalences is finitely based.

For the basics of universal algebra, the reader is referred to either [11] or [1].

In order to avoid writing too many parantheses in expressions involving nonassociative multiplication, we adopt the following conventions: $X \cdot Y$ stands for $(X)(Y)$, and missing parantheses are always assumed to be grouped to the left. So, for example, $x \cdot yzu = x((yz)u)$.

For equivalence relations, we will write $x \leftrightarrow y$ instead of $x \rightarrow y$. So, $x \leftrightarrow y$ iff $xy = x$ iff $yx = y$.

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1. Three-variable equations of equivalence algebras

We denote by $\mathbf{E}$ the variety generated by equivalence algebras.

Let us denote by $\mathbf{E}_0$ the variety of groupoids satisfying the following four equations:

(1) $xx = x$,
(2) $x \cdot yz = xy \cdot xz$,
(3) $xyx = x$,
(4) $xyzyx = xzyx$.

It is easy to check that these four equations are satisfied by any equivalence algebra. Consequently, the variety $\mathbf{E}$ is contained in $\mathbf{E}_0$.

Proposition 1.1. *The following equations are consequences of* (1),(2),(3)*:*

(5) $x \cdot yx = x$,
(6) $x \cdot xy = xy$,
(7) $xy \cdot y = xy$,
(8) $x \cdot xyz = x \cdot yz$,
(9) $xz \cdot yz = xz$,
(10) $xy \cdot zx = xyzx$.

Proof. (5) $x \cdot yx =_{(2)} xy \cdot xx =_{(1)} xy \cdot x =_{(3)} x$.
(6) $x \cdot xy =_{(5)} (x \cdot yx)(xy) =_{(2)} x \cdot yxy =_{(3)} xy$.
(7) $xy \cdot y =_{(5)} (xy)(y \cdot xy) =_{(4)} xy$.
(8) $x \cdot xyz =_{(2)} (x \cdot xy)(xz) =_{(6)} xy \cdot xz =_{(2)} x \cdot yz$.
(9) $xz \cdot yz =_{(2)} (xz \cdot y)(xz \cdot z) =_{(7)} (xz \cdot y)(xz) =_{(3)} xz$.
(10) $xy \cdot zx =_{(2)} (xy \cdot z)(xy \cdot x) =_{(3)} xyzx$. □

Lemma 1.2. *Let $A \in \mathbf{E}_0$ and a, b be two elements of A. Then $ab = a$ iff $ba = b$. Also, $ab = b$ iff $ba = a$.*

Proof. Use the equations (3) and (5). □

For two elements a, b of a groupoid $A \in \mathbf{E}_0$, we write $a \leftrightarrow b$ iff $ab = a$ iff $ba = b$.

Lemma 1.3. *Let $A \in \mathbf{E}_0$. Then $\leftrightarrow$ is a congruence of A. (Also, every equivalence on A extending $\leftrightarrow$ is a congruence of A.) We have $ab \leftrightarrow b$ for all $a, b \in A$.*

Proof. Reflexivity of $\leftrightarrow$ follows from (1). Symmetry and transitivity can be proved as follows. If $ab = a$, then $ba = b \cdot ab =_{(5)} b$. If $ab = a$ and $bc = b$, then $ac =_{(6)} a \cdot ac = a \cdot abc =_{(8)} a \cdot bc = ab = a$. So, $\leftrightarrow$ is an equivalence on A. By (7), $ab \leftrightarrow b$ for all $a, b \in A$. This can be used to prove that every equivalence on A extending $\leftrightarrow$ is a congruence. □

It follows that if F is a free algebra in $\mathbf{E}_0$ and $t \in F$, then $t \leftrightarrow x$, where x is the last variable in t. In particular, if t, u are two terms with the same last variables, then the equation $tu = t$ is satisfied in $\mathbf{E}_0$. This will be used extensively in the following.

Lemma 1.4. *The following equations are consequences of* (1),(2),(3),(4)*:*

(11) $xyzxy = xy$,
(12) $xyzyz = xyz$,
(13) $xyzyxy = xyzy$,
(14) $xyzx \cdot zy = x \cdot zy$,

(15) $x \cdot yz \cdot y = xyzy$,
(16) $xyz \cdot yx = xzy \cdot zx$,
(17) $xyzxz = xzyz$.

Proof. (11) $xy = xy \cdot xyzxy = xy \cdot xyz \cdot xyx \cdot xyy = xyzx \cdot xy = xyzxx \cdot xyzxy = xyzx \cdot xyzxy = xyzxy$.

(12) $xyzyz = xyzy \cdot xyzyz = xyzy \cdot xy \cdot xyzyz = xyzy \cdot xyz = xyz$.

(13) Where $a = xyzy$, we have $xyzy = ax \cdot a = axx \cdot axy \cdot axz \cdot axy = ax \cdot yzy = axy$.

(14) $xyzx \cdot zy = xyzxz \cdot xyzxy = (xyzxz \cdot xyz)(xyzxzx)(xyzxzy) = xyzxz \cdot xyzxzx \cdot xyzxzy = xyzxzx \cdot xyzxzy = (xy)zxz(xy) =_{(4)} (xy)xz(xy) = xz \cdot xy = x \cdot zy$.

(15) Put $a = x \cdot yz \cdot y$ and $b = xyzy$. We have $xzyx(x \cdot yz) =_{(9)} xzyx(xzyx \cdot yz) = xzyx \cdot yz =_{(14)} x \cdot yz$, so that $x \cdot yz \cdot xzyx = xzyx$ and hence

$$bx =_{(4)} xzyx = x \cdot yz \cdot xzyx = (x \cdot yz) \cdot xz \cdot (x \cdot yz \cdot y) \cdot (x \cdot yz \cdot x)$$
$$= (x \cdot yz)(x \cdot yz \cdot y)x = ax.$$

Also,

$$xa = xx \cdot (x \cdot yz) \cdot xy = x \cdot (x \cdot yz) \cdot xy = (x \cdot yz) \cdot xy = x \cdot yzy = xy,$$
$$xb = xx \cdot xy \cdot xz \cdot xy = x \cdot xy \cdot xz \cdot xy = xy \cdot xz \cdot xy = xy.$$

From $ax = bx$ and $xa = xb$ we get $a = ax \cdot xa = bx \cdot xb = b$.

(16) Put $a = xyz \cdot yx$ and $b = xzy \cdot zx$. We have

$$a \cdot xy = xy \cdot z \cdot yx \cdot xy =_{(4)} xy \cdot yx \cdot z \cdot yx \cdot xy = x \cdot z \cdot yx \cdot xy = xz \cdot yx \cdot xy$$
$$= xzy \cdot xzx \cdot xy = xzyx \cdot xy = xzyxx \cdot xzyxy = xzyx \cdot xzyxy = xzyxy,$$

so that

$$a \cdot xy \cdot xz = (xz)yxy(xz) =_{(4)} (xz)xy(xz) = xy \cdot xz = x \cdot yz.$$

Also, noting that for any u with $u \leftrightarrow x$, we know that $u \cdot xy = ux \cdot uy = u \cdot uy = uy$, we have

$$b \cdot xy \cdot xz = xzy \cdot zx \cdot xy \cdot xz = xz \cdot y \cdot zx \cdot y \cdot xz =_{(4)} xz \cdot zx \cdot y \cdot xz = xy \cdot xz = x \cdot yz.$$

This proves $a \cdot xy \cdot xz = b \cdot xy \cdot xz$. Since

$$xz \cdot (a \cdot xy) = xz \cdot xzyxy = xzx \cdot xzz \cdot xzy \cdot xzx \cdot xzy = x \cdot xz \cdot xzy \cdot x \cdot xzy$$
$$= xzy,$$
$$xz \cdot (b \cdot xy) = xz \cdot (xzy \cdot zx \cdot xy) = xzy \cdot (xz \cdot zx) \cdot (xz \cdot xy) = xzyx \cdot (x \cdot zy)$$
$$= xzyx \cdot (x \cdot xzy) = xzy,$$

we also get $xz \cdot (a \cdot xy) = xz \cdot (b \cdot xy)$. It follows that $a \cdot xy = b \cdot xy$. But we also have $xy \cdot a = xy \cdot b$, since $xy \cdot a = xyzx$ (easily) and

$$xy \cdot b = (xy \cdot xzy) \cdot (xy \cdot zx) = xy \cdot (xy \cdot zx) = xy \cdot zx = xyzx.$$

Now $a \cdot xy = b \cdot xy$ and $xy \cdot a = xy \cdot b$ yield $a = b$.

(17) We have

$$xyzxz \cdot x =_{(12)} xyzx =_{(4)} xzyz \cdot x,$$
$$x \cdot xyzxz = xx \cdot xy \cdot xz \cdot xx \cdot xz = xy \cdot xz \cdot x \cdot xz = x \cdot yz \cdot x \cdot xz = x \cdot xz = xz,$$
$$x \cdot xzyz = (x \cdot xz) \cdot xy \cdot xz = xz.$$

From $xyzxz \cdot x = xzyz \cdot x$ and $x \cdot xyzxz = x \cdot xzyz$ we get $xyzxz = xzyz$. □

Theorem 1.5. *The equations* (1)-(4) *imply all three-variable equations valid in equivalence algebras. The free algebra over* $\{x,y,z\}$ *in both* $\mathbf{E}$ *and* $\mathbf{E}_0$ *has 36 elements, represented by the terms*

$$x,\ xy,\ x\cdot yz,\ xyz,\ xyzy,\ xyzx,\ xyz\cdot yx$$

and all terms obtained from these by permutations of variables. (These 39 elements are pairwise different, except for $xyz\cdot yx = xzy\cdot zx$ *and the two symmetric cases.)*

Proof. Let us work in the free algebra F over $\{x,y,z\}$ in $\mathbf{E}_0$. Denote by Γ the set of the 36 elements listed above. (Their number is at most 36, since $xyz\cdot yx = xzy\cdot zx$ has been verified in Lemma 1.4.)

Note that if a,b are two terms in the three letters, and, say, $a \leftrightarrow x$, then the left distributive law allows us to write ab as a term $t(a,ay,az)$. Since a is a left unit and right zero in the subgroupoid generated by $\{a,ay,az\}$, it is easy to see that $t(a,ay,az)$ reduces to either a, or a term in ay and az. Using the left distributive law again, together with the two-variable equations, we see that ab is reduced to one of $a, ay, az, a\cdot yz, a\cdot zy$. Now using all of the equations (1)–(17) above, one can show with modest effort that whenever $a\in\Gamma$, then $au\in\Gamma$ if u is either a variable or a product of two variables. Thus it follows that $F=\Gamma$.

It is easy to see that the 36 elements of Γ represent pairwise different term functions on the three-element equivalence algebra with two equivalence blocks. Consequently, F is also free over $\{x,y,z\}$ in the variety $\mathbf{E}$. □

Proposition 1.6. *The equation* (4) *cannot be derived from* (1),(2),(3).

Proof. A 24-element groupoid A satisfying (1),(2) and (3) but not satisfying (4) can be constructed in the following way. Let $A=\{0,1\}\times\{0,1\}\times\{0,1\}\times\{0,1,2\}$. Let $a=(i,j,k,m)$ and $b=(i',j',k',n)$ be two elements of A. If $n=m$, we put $a\cdot b=a$. If $n\equiv m+1 \mod 3$, we put

$$a\cdot b=\begin{cases}(i',i,1-k,n) & \text{if } i=j=i'=0 \text{ and } (m,n)\in\{(0,1),(2,0)\},\\ (i',i,k,n) & \text{otherwise.}\end{cases}$$

If $m\equiv n+1 \mod 3$, we put

$$a\cdot b=\begin{cases}(j,j',1-k,n) & \text{if } i=j=j'=0 \text{ and } (m,n)\in\{(1,0),(0,2)\},\\ (j,j',k,n) & \text{otherwise.}\end{cases}$$

The verification of the equations is a bit tedious. The groupoid A does not satisfy any of the equations (14)–(17). □

2. Subdirectly irreducible algebras in $\mathbf{E}_0'$

This section contains some auxiliary results. We denote by $\mathbf{E}_0'$ the variety of groupoids determined by the equations (1), (2) and (3). By 1.6, $\mathbf{E}_0$ is properly contained in $\mathbf{E}_0'$.

For an algebra $A\in\mathbf{E}_0'$ and an element $a\in A$, define a binary relation $\sim_a$ on A by $x\sim_A y$ iff $ax=ay$. It follows from the left distributivity of A that $\sim_a$ is a congruence of A for any $a\in A$.

Lemma 2.1. *Let $A \in \mathbf{E}_0'$. The following are equivalent for an element $e \in A$:*

1. *$\sim_e$ is the identity;*
2. *$\{e\}$ is a block of $\leftrightarrow$;*
3. *$ex = ey$ implies $x = y$;*
4. *$ex = x$ for all $x \in A$;*
5. *$xe = e$ for all $x \in A$.*

Proof. (1) implies (2): Let $\sim_e$ be the identity. If $x \leftrightarrow e$, then $ex = e = ee$, so that $x \sim_e e$ and hence $x = e$.

(2) implies (1): Let $\{e\}$ be a block of $\leftrightarrow$ and let $x \sim_e y$, i.e., $ex = ey$. Since $xe \leftrightarrow e$, we have $xe = e$ and hence $ex = x$ by Lemma 1.2. Similarly, $ey = y$. From $ex = ey$ we get $x = y$.

(3) is a reformulation of (1), and (4) is equivalent to (5) by Lemma 1.2. (3) implies (4): $e(ex) = ex$. (5) implies (2): If $e \leftrightarrow x$, then $xe = x$, but $xe = e$ by assumption, so $x = e$. □

An element e of an algebra $A \in \mathbf{E}_0'$, satisfying the equivalent conditions of 2.1, will be called singular.

Proposition 2.2. *All isomorphism types of subdirectly irreducible algebras in $\mathbf{E}_0'$ (including the one-element algebra) come in pairs, each pair containing one algebra A without singular elements (or a one-element algebra), and one algebra A' resulting from A by adjoining a singular element.*

Proof. It is easy. Observe that if e is a singular element of an algebra $A \in \mathbf{E}_0'$, then $A - \{e\}$ is a subalgebra of A and for every congruence α of $A - \{e\}$, $\alpha \cup \mathrm{id}_A$ is a congruence of A. Also, if S is any set of singular elements of A, then $S^2 \cup \mathrm{id}_A$ is a congruence of A. This implies that if e is a singular element of a subdirectly irreducible algebra $A \in \mathbf{E}_0'$ of cardinality at least 3, then e is the only singular element of A and $A - \{e\}$ is a subdirectly irreducible subalgebra of A without a singular element. □

Lemma 2.3. *Let $A \in \mathbf{E}_0'$ be a subdirectly irreducible algebra with at least three elements and without singular elements; denote by θ the monolith of A. Then $x\theta y$ implies $x \leftrightarrow y$ and $ax = ay$ for all $a \in A$.*

Proof. If θ is not contained in $\leftrightarrow$, then $\leftrightarrow$ is the identity, $xy = x$ implies $x = y$, but from $x \cdot yx = x$ we get $yx = x$ for all $x, y \in A$; then A has only two elements, a contradiction. So, $x\theta y$ implies $x \leftrightarrow y$. The rest follows from Lemma 2.1. □

3. Main results

The equivalence algebra on $\{a, b, c\}$, corresponding to the equivalence with two blocks $\{a, b\}$ and $\{c\}$, will be denoted by E_3. This groupoid can be also given by its multiplication table:

	a	b	c
a	a	a	c
b	b	b	c
c	a	b	c

Theorem 3.1. *The variety generated by equivalence algebras is determined by the equations*

(E1) $xx = x$,
(E2) $x \cdot yz = xy \cdot xz$,
(E3) $xyx = x$,
(E4) $yzxyuz = yuz$,
(E5) $(u \cdot yzxy)z = uyz$.

The only subdirectly irreducible groupoids in the variety generated by equivalence algebras are (up to isomorphism) the groupoid E_3 and its subgroupoids.

Proof. It is easy to check that the five equations are satisfied by any equivalence algebra. For the rest, it is sufficient to show that any subdirectly irreducible algebra A satisfying the five equations can be embedded into E_3. According to 2.2, it is sufficient to consider just the case when A contains no singular elements.

For every element $c \in A$ define a binary relation β_c on A as follows: $(a, b) \in \beta_c$ if and only if $ab = a$ and $ac = bc$. Clearly, β_c is an equivalence. If $(a, b) \in \beta_c$ and $d \in A$, then $bcabdc = acabdc = abdc = adc$, while $bcabdc = bdc$ according to (E4), so that $adc = bdc$; also, $(d \cdot bcab)c = (d \cdot acab)c = (d \cdot ab)c = dac$, while $(d \cdot bcab)c = dbc$ according to (E5), and we get $dac = dbc$. This proves that β_c is a congruence of A.

Denote by μ the monolith of A. If there exists an element c with $\mu \subseteq \beta_c$, then for any $(x, y) \in \mu$ we have $xc = yc$, so that (according to Lemma 2.3) $x = xc \cdot x = xc \cdot y = yc \cdot y = y$. Since this is impossible and μ is the monolith, it follows that β_c is the identity for any $c \in A$. Hence, A satisfies the quasi-equation

$$(xy = x \;\&\; xz = yz) \Rightarrow x = y.$$

Now let a, b, c be any elements of A. Taking $x = a(bc), y = ac, z = a$, we conclude that $a(bc) = ac$. Since a, b, c are arbitrary, we also have

$$abc = ab \cdot ac = a \cdot bc = ac\,.$$

Thus A is a rectangular semigroup. Since A is subdirectly irreducible, it follows that A is a two-element semigroup satisfying either the law $xy = x$ or the law $xy = y$. In both cases, A is embeddable into E_3. □

Theorem 3.2. *The variety $\mathbf{E}_0$ is not locally finite; it contains an infinite, four-generated groupoid. Consequently, $\mathbf{E} \neq \mathbf{E}_0$. (The variety generated by equivalence algebras cannot be defined by three-variable equations only.)*

Proof. We are going to define an infinite groupoid G with the underlying set $\{a_i : i \in Z\} \cup \{b_i : i \in Z\}$ (these are pairwise distinct elements). Denote by F_{01} the set of the integers congruent with either 0 or 1 modulo 4, and put $F_{23} = Z - F_{01}$. Denote by $\equiv$ the equivalence with the two blocks F_{01} and F_{23} on Z.

For $i, j \in Z$ put $a_i a_j = a_i$, $b_i b_j = b_i$,

$$a_i b_j = \begin{cases} b_{i+1} & \text{if } i \equiv j - 1, \\ b_{i-1} & \text{if } i \equiv j + 1, \end{cases} \qquad b_i a_j = \begin{cases} a_{i+1} & \text{if } i \equiv j - 1, \\ a_{i-1} & \text{if } i \equiv j + 1. \end{cases}$$

One can easily check that this idempotent groupoid satisfies $x \cdot yx = xy \cdot x = x$. Also, it is easy to check that $a_i \cdot a_j b_k = a_i b_k$ for all i, j, k. Then it is not difficult to verify the left distributive law. The equation (4) is satisfied in any groupoid from $\mathbf{E}_0'$ which has just two blocks of $\leftrightarrow$, so $G \in \mathbf{E}_0$. The groupoid is generated by $\{a_0, b_0, a_1, b_1\}$. According to Theorem 3.1, the variety $\mathbf{E}$ is generated by the finite groupoid E_3, so that it is locally finite; on the other hand, $\mathbf{E}_0$ is not locally finite. □

Also, the eight-element groupoid G_8 with elements a, b, c, d, e, f, g, h and multiplication table

	a	b	c	d	e	f	g	h
a	a	a	a	a	e	e	h	h
b	b	b	b	b	f	f	g	g
c	c	c	c	c	e	e	g	g
d	d	d	d	d	f	f	h	h
e	a	a	c	c	e	e	e	e
f	b	b	d	d	f	f	f	f
g	b	b	c	c	g	g	g	g
h	a	a	d	d	h	h	h	h

belongs to $\mathbf{E}_0 - \mathbf{E}$.

There is a more general construction for these and similar examples. Choose any non-void set C and put $A = C \times \{0, 1, 2, 3\}$. Now choose permutations $\pi_{02}, \pi_{03}, \pi_{12}, \pi_{13}$ of C. Define $(c, i) \cdot (d, j) = (c, i)$ if either $\{i, j\} \subseteq \{0, 1\}$ or $\{i, j\} \subseteq \{2, 3\}$. If $i \in \{0, 1\}$ and $j \in \{2, 3\}$, put $(c, i) \cdot (d, j) = (\pi_{ij}(c), j)$. And if $i \in \{2, 3\}$ and $j \in \{0, 1\}$, put $(c, i) \cdot (d, j) = (\pi_{ji}^{-1}(c), j)$. This defines a groupoid A in which every three-generated subgroupoid is either a rectangular band or else a subdirect product of a two- or three-element right-zero semigroup and the groupoid E_3. With the appropriate choice of the permutations π_{ij} we can ensure that A has an infinite four-generated subgroupoid—for example, by having $\pi_{02}\pi_{12}^{-1}\pi_{13}\pi_{03}^{-1}$ be a permutation of infinite order containing an infinite cycle.

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Charles University, Sokolovská 83, 186 00 Praha 8, Czech Republic

Vanderbilt University, Dept. of Mathematics, Nashville, Tennessee 37240