THREE-VARIABLE EQUATIONS OF POSETS

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Abstract. We find an independent base for three-variable equations of posets.

1. Introduction

For a poset $(P, \leq)$ we can define multiplication on P by $xy = x$ if $x \leq y$, and $xy = y$ otherwise. The groupoid $(P, \cdot)$ will be called the poset groupoid of $(P, \leq)$. Since the correspondence between posets and poset groupoids is one-to-one, posets will be sometimes identified with their poset groupoids.

More generally, one can identify an arbitrary directed graph with loops (a set with a reflexive binary relation) with a groupoid defined in a similar way, and then employ methods of universal algebra to investigate various interesting classes of directed graphs. This approach was started in [5] and continued by various authors in the case of the class of tournaments. The variety generated by tournaments has been investigated in [1] and [2]. In [3], we investigate in a similar way the variety generated by equivalence relations. The variety generated by posets (partially ordered sets) seems to be the most natural and interesting candidate in this respect. In the present paper we start to investigate this variety.

We denote by $\mathbf{P}$ the variety generated by posets (or poset groupoids). For any $n \geq 1$, let $\mathbf{P}_n$ denote the variety generated by all n-element posets, and let $\mathbf{P}^n$ denote the variety determined by the at most n-variable equations of posets. So, $\mathbf{P}_n \subseteq \mathbf{P}_{n+1} \subseteq \mathbf{P} \subseteq \mathbf{P}^{n+1} \subseteq \mathbf{P}^n$ for all n. It is easy to see that the free groupoid on n generators in $\mathbf{P}$ is a free groupoid on n generators in $\mathbf{P}_n$, as well as in $\mathbf{P}^n$. The aim of this paper is to describe the free groupoid on three generators in $\mathbf{P}$, and to find an independent base for the equations of the variety $\mathbf{P}^3$. We leave as an open problem the question whether the variety $\mathbf{P}$ is finitely based.

2. A base for the equations of $\mathbf{P}^3$

It is easy to check that every poset groupoid satisfies the following five equations:

(1) $xx = x$,
(2) $xy \cdot x = yx$,

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(3) $xy \cdot y = xy$,
(4) $x(xy \cdot z) = x \cdot yz$,
(5) $(xy \cdot z)y = xz \cdot y$.

The following two-variable equations are consequences of (1)–(5):

(6) $x \cdot xy = xy$. Proof: $x \cdot xy =_{(3)} x(xy \cdot y) =_{(4)} x \cdot yy =_{(1)} xy$.
(7) $x \cdot yx = yx$. Proof: $x \cdot yx =_{(2)} (yx \cdot x) \cdot yx =_{(3)} yx \cdot yx =_{(1)} yx$.
(8) $xy \cdot yx = yx$. Proof: $xy \cdot yx =_{(2)} (yx \cdot y) \cdot yx =_{(2)} y \cdot yx =_{(6)} yx$.

For a groupoid $A \in \mathbf{P}^3$ put $x \leq y$ if and only if $xy = x$. We are now able to show that this relation is an ordering on P. Reflexivity is clear. If $x \leq y$ and $y \leq z$, then $xz =_{(6)} x \cdot xz = x(xy \cdot z) =_{(4)} x \cdot yz = xy = x$, i.e., $x \leq z$. If $x \leq y$ and $y \leq z$, then $x = xy =_{(2)} yx \cdot y = yy =_{(1)} y$.

We continue with a list of three-variable consequences of (1)–(5):

(9) $xy \cdot xz = y \cdot xz$. Proof: $xy \cdot xz =_{(5)} ((x \cdot xz)y) \cdot xz =_{(6)} (xz \cdot y) \cdot xz =_{(2)} y \cdot xz$.
(10) $xy \cdot yz = xy \cdot z$. Proof: $xy \cdot yz =_{(4)} xy \cdot ((xy \cdot y)z) =_{(3)} xy \cdot (xy \cdot z) =_{(6)} xy \cdot z$.
(11) $xy \cdot zx = y \cdot zx$. Proof: $xy \cdot zx =_{(2)} (zx \cdot xy) \cdot zx =_{(10)} (zx \cdot y) \cdot zx =_{(2)} y \cdot zx$.
(12) $(x \cdot yz)y = xz \cdot y$. Proof: $(x \cdot yz)y =_{(5)} (xy \cdot yz)y =_{(10)} (xy \cdot z)y =_{(5)} xz \cdot y$.
(13) $(x \cdot yz)z = x \cdot yz$. Proof: $x \cdot yz \leq_{(3)} yz \leq_{(3)} z$; use the transitivity of $\leq$.
(14) $xy \cdot zy = x \cdot zy$. Proof: $xy \cdot zy =_{(5)} ((x \cdot zy)y) \cdot zy =_{(13)} (x \cdot zy) \cdot zy =_{(3)} x \cdot zy$.
(15) $(xy \cdot z)x = yz \cdot x$. Proof: $(xy \cdot z)x =_{(3)} x(xy \cdot z) \cdot x =_{(4)} (x \cdot yz)x =_{(3)} yz \cdot x$.
(16) $x(y \cdot xz) = y \cdot xz$. Proof: Since $x(y \cdot xz) \leq y \cdot xz \leq xz \leq z$, we have $x(y \cdot xz) = (x(y \cdot xz))z =_{(5)} (xz \cdot (y \cdot xz))z =_{(7)} (y \cdot xz)z =_{(13)} y \cdot xz$.
(17) $x(yx \cdot z) = yx \cdot z$. Proof: $x(yx \cdot z) =_{(10)} x(yx \cdot xz) =_{(16)} yx \cdot xz =_{(10)} yx \cdot z$.
(18) $x(y \cdot zx) = y \cdot zx$. Proof: $x(y \cdot zx) =_{(2)} (y \cdot zx)x \cdot (y \cdot zx) =_{(13)} (y \cdot zx)(y \cdot zx) =_{(1)} y \cdot zx$.
(19) $xy \cdot (yx \cdot z) = y \cdot xz$. Proof: $xy \cdot (yx \cdot z) =_{(2)} xy \cdot ((xy \cdot x)z) =_{(4)} xy \cdot xz =_{(9)} y \cdot xz$.
(20) $xy \cdot (xz \cdot y) = x \cdot zy$. Proof: $xy \cdot (xz \cdot y) =_{(14)} x(xz \cdot y) =_{(4)} x \cdot zy$.
(21) $xy \cdot (zx \cdot y) = zx \cdot y$. Proof: $xy \cdot (zx \cdot y) =_{(14)} x(zx \cdot y) =_{(17)} zx \cdot y$.
(22) $xy \cdot (yz \cdot x) = y \cdot zx$. Proof: $xy \cdot (yz \cdot x) =_{(11)} y(yz \cdot x) =_{(4)} y \cdot zx$.
(23) $xy \cdot (zy \cdot x) = zy \cdot x$. Proof: $xy \cdot (zy \cdot x) =_{(11)} y(zy \cdot x) =_{(17)} zy \cdot x$.
(24) $xy \cdot (x \cdot yz) = x \cdot yz$. Proof: $xy \cdot (x \cdot yz) =_{(9)} y(x \cdot yz) =_{(16)} x \cdot yz$.
(25) $xy \cdot (x \cdot zy) = x \cdot zy$. Proof: $xy \cdot (x \cdot zy) =_{(9)} y(x \cdot zy) =_{(18)} x \cdot zy$.
(26) $xy \cdot (y \cdot xz) = y \cdot xz$. Proof: $xy \cdot (y \cdot xz) =_{(10)} xy \cdot xz =_{(9)} y \cdot xz$.
(27) $xy \cdot (y \cdot zx) = y \cdot zx$. Proof: $xy \cdot (y \cdot zx) =_{(10)} xy \cdot zx =_{(11)} y \cdot zx$.

(28) $xy \cdot (z \cdot yx) = z \cdot yx$. Proof: $xy \cdot (z \cdot yx) =_{(8)} xy \cdot (z(xy \cdot yx)) =_{(16)} z(xy \cdot yx) =_{(8)} z \cdot yx$.
(29) $(xy \cdot z) \cdot yx = z \cdot yx$. Proof: $(xy \cdot z) \cdot yx =_{(8)} (xy \cdot z)(xy \cdot yx) =_{(9)} z(xy \cdot yx) =_{(8)} z \cdot yx$.
(30) $(xy \cdot z) \cdot xz = y \cdot xz$. Proof: $(xy \cdot z) \cdot xz =_{(14)} xy \cdot xz =_{(9)} y \cdot xz$.
(31) $(xy \cdot z) \cdot zx = yz \cdot x$. Proof: $(xy \cdot z) \cdot zx =_{(10)} (xy \cdot z)x =_{(15)} yz \cdot x$.
(32) $(xy \cdot z) \cdot yz = xy \cdot z$. Proof: $(xy \cdot z) \cdot yz =_{(14)} xy \cdot yz =_{(10)} xy \cdot z$.
(33) $(xy \cdot z) \cdot zy = xz \cdot y$. Proof: $(xy \cdot z) \cdot zy =_{(10)} (xy \cdot z)y =_{(5)} xz \cdot y$.
(34) $(x \cdot yz) \cdot xy = z \cdot xy$. Proof: $(x \cdot yz) \cdot xy =_{(9)} yz \cdot xy =_{(11)} z \cdot xy$.
(35) $(x \cdot yz) \cdot yx = z \cdot yx$. Proof: $(x \cdot yz) \cdot yx =_{(11)} yz \cdot yx =_{(9)} z \cdot yx$.
(36) $(x \cdot yz) \cdot xz = y \cdot xz$. Proof: $(x \cdot yz) \cdot xz =_{(9)} yz \cdot xz =_{(14)} y \cdot xz$.
(37) $(x \cdot yz) \cdot zx = yz \cdot x$. Proof: $(x \cdot yz) \cdot zx =_{(11)} yz \cdot zx =_{(10)} yz \cdot x$.
(38) $(x \cdot yz) \cdot zy = xz \cdot y$. Proof: $(x \cdot yz) \cdot zy =_{(2)} (x \cdot yz)(yz \cdot y) =_{(10)} (x \cdot yz)y =_{(12)} xz \cdot y$.
(39) $(xy \cdot z)(yx \cdot z) = y \cdot xz$. Proof: $(xy \cdot z)(yx \cdot z) =_{(14)} xy \cdot (yx \cdot z) =_{(19)} y \cdot xz$.
(40) $(xy \cdot z)(xz \cdot y) = xz \cdot y$. Proof: $(xy \cdot z)(xz \cdot y) =_{(33)} ((xz \cdot y) \cdot yz)(xz \cdot y) =_{(2)} yz \cdot (xz \cdot y) =_{(23)} xz \cdot y$.
(41) $(xy \cdot z)(zx \cdot y) = z \cdot xy$. Proof: $(xy \cdot z)(zx \cdot y) =_{(10)} (xy \cdot z)(zx \cdot xy) =_{(11)} z(zx \cdot xy) =_{(10)} z(zx \cdot y) =_{(4)} z \cdot xy$.
(42) $(xy \cdot z)(yz \cdot x) = yz \cdot x$. Proof: $(xy \cdot z)(yz \cdot x) =_{(10)} (xy \cdot yz)(yz \cdot x) =_{(10)} (xy \cdot yz)x =_{(10)} (xy \cdot z)x =_{(15)} yz \cdot x$.
(43) $(xy \cdot z)(zy \cdot x) = z \cdot yx$. Proof: $(xy \cdot z)(zy \cdot x) =_{(23)} (xy \cdot z)(xy \cdot (zy \cdot x)) =_{(9)} z(xy \cdot (zy \cdot x)) =_{(23)} z(zy \cdot x) =_{(4)} z \cdot yx$.
(44) $(xy \cdot z)(x \cdot yz) = x \cdot yz$. Proof: $(xy \cdot z)(x \cdot yz) =_{(4)} (xy \cdot z) \cdot x(xy \cdot z) =_{(7)} x(xy \cdot z) =_{(4)} x \cdot yz$.
(45) $(xy \cdot z)(x \cdot zy) = x \cdot zy$. Proof: $(xy \cdot z)(x \cdot zy) =_{(20)} (xy \cdot z)(xy \cdot (xz \cdot y)) =_{(9)} z(xy \cdot (xz \cdot y)) =_{(20)} z(x \cdot zy) =_{(16)} x \cdot zy$.
(46) $(xy \cdot z)(y \cdot xz) = y \cdot xz$. Proof: $(xy \cdot z)(y \cdot xz) =_{(9)} (xy \cdot z)(xy \cdot xz) =_{(9)} z(xy \cdot xz) =_{(9)} z(y \cdot xz) =_{(18)} y \cdot xz$.
(47) $(xy \cdot z)(y \cdot zx) = y \cdot zx$. Proof: $(xy \cdot z)(y \cdot zx) =_{(11)} (xy \cdot z)(xy \cdot zx) =_{(9)} z(xy \cdot zx) =_{(11)} z(y \cdot zx) =_{(16)} y \cdot zx$.
(48) $(xy \cdot z)(z \cdot yx) = z \cdot yx$. Proof: $(xy \cdot z)(z \cdot yx) =_{(10)} (xy \cdot z) \cdot yx =_{(29)} z \cdot yx$.
(49) $(x \cdot yz)(xy \cdot z) = x \cdot yz$. Proof: $(x \cdot yz)(xy \cdot z) =_{(4)} x(xy \cdot z) \cdot (xy \cdot z) =_{(3)} x(xy \cdot z) =_{(4)} x \cdot yz$.
(50) $(x \cdot yz)(yx \cdot z) = y \cdot xz$. Proof: $(x \cdot yz)(yx \cdot z) =_{(19)} (yx \cdot (xy \cdot z))(yx \cdot z) =_{(9)} (xy \cdot z)(yx \cdot z) =_{(39)} y \cdot xz$.
(51) $(x \cdot yz)(xz \cdot y) = xz \cdot y$. Proof: $(x \cdot yz)(xz \cdot y) =_{(20)} (xz \cdot (xy \cdot z))(xz \cdot y) =_{(9)} (xy \cdot z)(xz \cdot y) =_{(40)} xz \cdot y$.
(52) $(x \cdot yz)(zx \cdot y) = z \cdot xy$. Proof: $(x \cdot yz)(zx \cdot y) =_{(22)} (zx \cdot (xy \cdot z))(zx \cdot y) =_{(9)} (xy \cdot z)(zx \cdot y) =_{(41)} z \cdot xy$.
(53) $(x \cdot yz)(zy \cdot x) = z \cdot yx$. Proof: $(x \cdot yz)(zy \cdot x) =_{(11)} yz \cdot (zy \cdot x) =_{(19)} z \cdot yx$.
(54) $(x \cdot yz)(x \cdot zy) = x \cdot zy$. Proof: $(x \cdot yz)(x \cdot zy) =_{(9)} yz \cdot (x \cdot zy) =_{(28)} x \cdot zy$.

(55) $(x \cdot yz)(y \cdot xz) = y \cdot xz$. Proof: $(x \cdot yz)(y \cdot xz) =_{(20)} (xz \cdot (xy \cdot z))(y \cdot xz) =_{(11)} (xy \cdot z)(y \cdot xz) =_{(46)} y \cdot xz$.

(56) $(x \cdot yz)(y \cdot zx) = y \cdot zx$. Proof: $(x \cdot yz)(y \cdot zx) =_{(22)} (zx \cdot (xy \cdot z))(y \cdot zx) =_{(11)} (xy \cdot z)(y \cdot zx) =_{(47)} y \cdot zx$.

(57) $(x \cdot yz)(z \cdot xy) = z \cdot xy$. Proof: $(x \cdot yz)(z \cdot xy) =_{(4)} x(xy \cdot z) \cdot (z \cdot xy) =_{(38)} xz \cdot xy =_{(9)} z \cdot xy$.

(58) $(x \cdot yz)(z \cdot yx) = z \cdot yx$. Proof: $(x \cdot yz)(z \cdot yx) =_{(4)} x(xy \cdot z) \cdot (z \cdot yx) =_{(43)} x(xy \cdot z) \cdot (xy \cdot z)(zy \cdot x) =_{(10)} x(xy \cdot z) \cdot (zy \cdot x) =_{(4)} (x \cdot yz)(zy \cdot x) =_{(11)} yz \cdot (zy \cdot x) =_{(19)} z \cdot yx$.

Theorem 1. *The equations* (1)–(5) *constitute a base for the equational theory of* $\mathbf{P}^3$. *The free groupoid on three generators* x, y, z *in* $\mathbf{P}$ *has 21 elements*

$a = x$	$h = yz$	$o = zy \cdot x$
$b = y$	$i = zy$	$p = x \cdot yz$
$c = z$	$j = xy \cdot z$	$q = x \cdot zy$
$d = xy$	$k = yx \cdot z$	$r = y \cdot xz$
$e = yx$	$l = xz \cdot y$	$s = y \cdot zx$
$f = xz$	$m = zx \cdot y$	$t = z \cdot xy$
$g = zx$	$n = yz \cdot x$	$u = z \cdot yx$

and its multiplication table is shown below.

	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u
a	a	d	f	d	e	f	g	p	q	p	k	q	m	n	o	p	q	r	s	t	u
b	e	b	h	d	e	r	s	h	i	j	r	l	m	s	o	p	q	r	s	t	u
c	g	i	c	t	u	f	g	h	i	j	k	l	t	n	u	p	q	r	s	t	u
d	e	d	j	d	e	r	s	j	q	j	r	q	m	s	o	p	q	r	s	t	u
e	e	d	k	d	e	k	s	p	q	p	k	q	m	s	o	p	q	r	s	t	u
f	g	l	f	t	u	f	g	p	l	p	k	l	t	n	u	p	q	r	s	t	u
g	g	m	f	m	u	f	g	p	q	p	k	q	m	n	u	p	q	r	s	t	u
h	n	i	h	t	u	r	n	h	i	j	r	l	t	n	u	p	q	r	s	t	u
i	o	i	h	t	o	r	s	h	i	j	r	l	t	s	o	p	q	r	s	t	u
j	n	l	j	t	u	r	n	j	l	j	r	l	t	n	u	p	q	r	s	t	u
k	n	l	k	t	u	k	n	p	l	p	k	l	t	n	u	p	q	r	s	t	u
l	o	l	j	t	o	r	s	j	l	j	r	l	t	s	o	p	q	r	s	t	u
m	o	m	j	m	o	r	s	j	q	j	r	q	m	s	o	p	q	r	s	t	u
n	n	m	k	m	u	k	n	p	q	p	k	q	m	n	u	p	q	r	s	t	u
o	o	m	k	m	o	k	s	p	q	p	k	q	m	s	o	p	q	r	s	t	u
p	n	l	p	t	u	r	n	p	l	p	r	l	t	n	u	p	q	r	s	t	u
q	o	q	j	t	o	r	s	j	q	j	r	q	t	s	o	p	q	r	s	t	u
r	n	l	r	t	u	r	n	p	l	p	r	l	t	n	u	p	q	r	s	t	u
s	s	m	k	m	u	k	s	p	q	p	k	q	m	s	u	p	q	r	s	t	u
t	o	t	j	t	o	r	s	j	q	j	r	q	t	s	o	p	q	r	s	t	u
u	u	m	k	m	u	k	s	p	q	p	k	q	m	s	u	p	q	r	s	t	u

Proof. Using the above listed 58 equations, it is easy to build the multiplication table of the free groupoid on three generators x, y, z. In order to see that the 21 elements are pairwise different, it is possible to check that the 21 terms behave differently on three-element posets. □

3. Single-exception differences from poset groupoids

Groupoids that do not differ much from poset groupoids can be conveniently described by a list of the exceptions $ab = c$ from the poset groupoid multiplication. Given a poset P and a triple of elements a, b, c of P, we denote by $P[ab = c]$ the groupoid with multiplication defined by

$$xy = \begin{cases} c \text{ if } (x, y) = (a, b), \\ x \text{ if } x \leq y \text{ and } (x, y) \neq (a, b), \\ y \text{ if } x \not\leq y \text{ and } (x, y) \neq (a, b). \end{cases}$$

Similarly as in the case of tournaments ([1] and [2]), these single-exception non-poset groupoids may turn out to be crucial for the investigation of the equational theory of poset groupoids.

Theorem 2. *Let P be a poset and let a, b, c be a triple of pairwise different elements of P such that a is incomparable with b. The groupoid $P[ab = c]$ belongs to $\mathbf{P}^3$ if and only if the following conditions are satisfied:*

(c1) *$c < b$ and $c \not\leq a$,*
(c2) *for any $x \in P$, $x > c$ implies $x \geq b$,*
(c3) *for any $x \in P$, $x < a$ implies $x < c$,*
(c4) *for any $x \in P$, $x > a$ if and only if $x > b$.*

Proof. First assume that $P[ab = c]$ belongs to $\mathbf{P}^3$. By (3) we have $c \leq b$, and hence $c < b$. If $c \leq a$, then $ba =_{(2)} ab \cdot a = ca = c \notin \{a, b\}$, a contradiction.

Suppose $x > c$ and $x \not\geq b$. Then $bx = x$ and we have $ax = a \cdot bx =_{(4)} a(ab \cdot x) = a \cdot cx = ac = c$, a contradiction.

Let $x < a$. If $x \not\leq b$, then $b = xb =_{(6)} x \cdot xb = xa \cdot xb =_{(9)} a \cdot xb = ab = c$, a contradiction. Hence $x \leq b$, and we get $xc = x \cdot ab =_{(4)} x(xa \cdot b) = x \cdot xb =_{(6)} xb = x$, so that $x \leq c$ and then $x < c$.

Suppose $x > a$ and $x \not\geq b$. Then $b = xb = cx \cdot b = (ab \cdot x)b =_{(5)} ax \cdot b = ab = c$, a contradiction.

Suppose $x > b$ and $x \not\geq a$. Then $c = cx = ab \cdot ax =_{(9)} b \cdot ax = bx = b$, a contradiction.

Conversely, it is easy to prove (1)–(5) under the assumption that (c1)–(c4) are satisfied. □

Theorem 3. *The equations* (1)–(5) *are independent.*

Proof. The first equation cannot be derived from the other ones, since it is the only absorption equation among them. Similarly, the second equation is the single equation among (1)–(5) with the property that the first variables of the left and of the right side are not the same.

Let P_1 be the three-element antichain with elements a, b, c. It is easy to check that the groupoid $P_1[ab = c]$ satisfies all the equations (1)–(5) but (3).

Let P_2 be the four-element poset with elements a, b, c, d, where the order relation is the reflexive and transitive closure of $a < d$, $c < d$, $c < b$. It is easy to check that the groupoid $P_2[ab = c]$ satisfies all the equations (1)–(5) but (4).

Let P_3 be the four-element poset with elements a, b, c, d, where the order relation is the reflexive and transitive closure of $a < d$, $c < b$. It is easy to check that the groupoid $P_3[ab = c]$ satisfies all the equations (1)–(5) but (5). □

References

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