ONE-ELEMENT EXTENSIONS IN THE VARIETY GENERATED BY TOURNAMENTS

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Abstract. We investigate congruences in one-element extensions of algebras in the variety generated by tournaments.

0. Introduction

Recently M. Maróti proved that every subdirectly irreducible algebra in the variety $\mathcal{T}$ generated by tournaments is a tournament; equivalently, the variety generated by tournaments coincides with the quasivariety generated by tournaments. This has been a conjecture formulated in the paper [3]; in that paper and in [1] we have proved some particular cases. In [3] we have also formulated a stronger conjecture, which remains open: A groupoid belongs to the variety $\mathcal{T}$ if and only if it satisfies the three-variable equations of tournaments and avoids the algebras $\mathbf{J}_3$ and $\mathbf{M}_n$ ($n \geq 3$; these algebras are defined below). This has been verified for all groupoids with at most ten elements.

The aim of this paper is to investigate one-element extensions in the variety $\mathcal{T}$. Let A and B be two groupoids such that $B \in \mathcal{T}$ and B is an extension of A by an element e. Denote by V the set of the elements $a \in A$ such that $a \to e$ in B. The main result of this paper states that the congruence of B generated by all pairs of incomparable elements from V has all nontrivial blocks contained in V. Since there is a hope that this could be useful for the solution of the stronger conjecture, we will formulate and prove this result in terms of algebras satisfying the three-variable equations of tournaments and avoiding $\mathbf{J}_3$ and $\mathbf{M}_n$. (See Theorem 2.12.)

For the terminology and notation see [4] and [2].

We denote by $\mathbf{T}$ the class of tournaments, and by $\mathcal{T}$ the variety generated by $\mathbf{T}$. For any $n \geq 1$, let $\mathcal{T}_n$ denote the variety generated by all n-element tournaments, and let $\mathcal{T}^n$ denote the variety determined by the at most n-variable equations of tournaments. So, $\mathcal{T}_n \subseteq \mathcal{T}_{n+1} \subseteq \mathcal{T} \subseteq \mathcal{T}^{n+1} \subseteq \mathcal{T}^n$ for all n.

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For a variety V and a positive integer n, we denote by $\mathbf{F}_n(V)$ the free algebra in V on n generators. According to Theorem 3 of [3], $\mathbf{F}_n(\mathcal{T}) = \mathbf{F}_n(\mathcal{T}_n) = \mathbf{F}_n(\mathcal{T}^n)$.

According to [3], the following four equations are a base for the equational theory of $\mathcal{T}^3$:

(e1) $xx = x$,
(e2) $xy = yx$,
(e3) $xy \cdot x = xy$,
(e4) $(xy \cdot xz)(xy \cdot yz) = xyz$

and the following are consequences of these four equations:

(e5) $(xy \cdot xz)x = xy \cdot xz$,
(e6) $(xy \cdot xz) \cdot yz = xyzy$,
(e7) $xyzy = xzyz$,
(e8) $(yzx)(xy \cdot xz) = xy \cdot xz$,
(e9) $xzyxz = xyz$.

According to Lemma 5 of [3], for any three elements a, b, c of an algebra $A \in \mathcal{T}^3$ we have:

(p1) If $ab \to c$, then a, b, c generate a semilattice.
(p2) If $ab \to c \to a$, then $bc = ab$.
(p3) If $a \to c \to ab$, then $c \to b$.
(p4) If $a \to c$ and $b \to c$, then $ab \to c$.
(p5) If $a \to c \to b$ and a, b, c, ab are four distinct elements, then the subgroupoid generated by a, b, c either contains just these four elements and $c \to ab$, or else it contains precisely five elements $a, b, c, ab, ab \cdot c$ and $a \to ab \cdot c \to b$.

Our proof in [2] of the fact that the variety $\mathcal{T}$ is not finitely based relied on an infinite sequence $\mathbf{M}_n$ $(n \geq 3)$ of algebras with the following properties: $\mathbf{M}_n$ is subdirectly irreducible, $|\mathbf{M}_n| = n + 2$ and $\mathbf{M}_n \in \mathcal{T}^n - \mathcal{T}^{n+1}$. These algebras are defined as follows. $\mathbf{M}_n = \{a, c, c, d_1, \dots, d_{n-2}, e\}$;

$$ab = e,$$
$$e \to a \to c,$$
$$e \to b \to c,$$
$$e \to c,$$
$$a \to d_1 \to d_2 \to \cdots \to d_{n-2} \to b,$$
$$d_i \to c \text{ for } i < n-2,$$
$$c \to d_{n-2},$$
$$d_i \to e \text{ for all } i,$$
$$d_i \to a \text{ for } i > 1,$$
$$d_i \to b \text{ for all } i,$$
$$d_j \to d_i \text{ for } j > i+1.$$

We will also need the five-element subdirectly irreducible algebra $\mathbf{J}_3 \in \mathcal{T}^3$, introduced in [3] and defined on $\{a, b, c, d, e\}$ by $a \to d \to b \to c \to a$, $c \to e$, $d \to c$, $d \to e$ and $ab = e$. The algebras $\mathbf{M}_3$, $\mathbf{M}_4$ and $\mathbf{J}_3$ are pictured in Fig. 1. (The monolith of $\mathbf{M}_n$ identifies ab with b; the monolith of $\mathbf{J}_3$ identifies ab with b with c.)

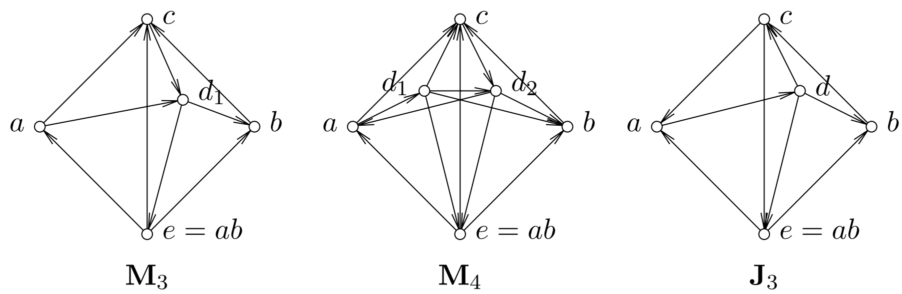

Fig. 1

Two elements a, b of an algebra $A \in \mathcal{T}^3$ are said to be comparable if either $a \to b$ or $b \to a$; we write $a \updownarrow b$ in that case. If a, b are incomparable, we write $a||b$.

We say that an algebra A avoids an algebra B if A contains no subalgebra isomorphic to B. We denote by $\mathcal{T}^*$ the class of the algebras belonging to $\mathcal{T}^3$ and avoiding the algebras $\mathbf{J}_3$ and $\mathbf{M}_n$ for all $n \geq 3$.

1. One-element extensions

Throughout this paper let A be an algebra belonging to $\mathcal{T}^*$; let $A = U \cup V$ be a partition of A into two disjoint subgroupoids such that $u \in U$, $v \in V$ and $u||v$ imply $uv \in U$; let e be an element not belonging to A; define an algebra B with the underlying set $A \cup \{e\}$ in such a way that A is a subgroupoid and $v \to e \to u$ for all $u \in U$ and $v \in V$. Then, as it is easy to see, B belongs to $\mathcal{T}^3$. We will assume that B avoids $\mathbf{J}_3$ and $\mathbf{M}_n$ for all $n \geq 3$, so that $B \in \mathcal{T}^*$.

1.1. **Proposition.** *The following are true:*

(1) *There are no elements $u \in U$, $v \in V$ and $a \in A$ with $u||v$, $u \to a \to v$ and $a \to uv$.*
(2) *There are no elements $u \in U$ and $v, w \in V$ with $u||v$, $u \to w$ and $v \to w$.*
(3) *There are no elements $u \in U$ and $v_1, v_2 \in V$ with $v_1||v_2$, $v_1 \to u \to v_2$ and $u \to v_1v_2$.*

Proof. Suppose there are such elements.

(1) Since $u \to a \to v \to e \to u$, $a \to uv$, $e \to uv$ and $a \updownarrow e$, these five elements constitute a subalgebra isomorphic to $\mathbf{J}_3$ (no matter whether $a \to e$ or $e \to a$).

(2) The elements $v \to e \to u$ with uv and w constitute a subalgebra isomorphic to $\mathbf{M}_3$.

(3) The elements $v_1 \to u \to v_2$ with v_1v_2 and e constitute a subalgebra isomorphic to $\mathbf{M}_3$.

We get a contradiction in each case. □

1.2. **Proposition.** *Let $u \in U$, $v \in V$, $u||v$. Then there is no element $a \in A$ with $u \to a \to v$.*

Proof. Suppose there is. Put $a' = uva$. By (p5) we have $u \to a' \to v$. Since $a' \to uv$, we get a contradiction with 1.1(1). □

1.3. **Proposition.** *Let $u \in U$, $v \in V$, $u||v$. Then there is no element $w \in V$ with $u \to w$.*

Proof. Suppose there is. By 1.1(2), $v \not\to w$. By 1.2, $w \not\to v$. Hence $v||w$. If $vw||u$, we get a contradiction with 1.1(2), since $u \to w$ and $vw \to w$. If $u \to vw$, we get a contradiction with 1.2, since $u \to vw \to v$. Hence $vw \to u$. Then also $vw \to uv$. We have $uvw = vuw = vwuvw = vwvw = vw$. Clearly, $vw \neq uv$ and $vw \neq w$. Hence $uv||w$. But then $uvw \in U$, a contradiction with $uvw = vw \in V$. □

For $v_1, v_2 \in V$ we write $v_1 \equiv v_2$ if for every $u \in U$, one of the following three cases takes place:

(1) $u \to v_1$ and $u \to v_2$;
(2) $v_1 \to u$ and $v_2 \to u$;
(3) $u||v_1$, $u||v_2$ and $uv_1 = uv_2$.

Clearly, $\equiv$ is an equivalence on V.

1.4. **Proposition.** *Let $v_1, v_2 \in V$, $v_1||v_2$. Then $v_1 \equiv v_2 \equiv v_1v_2$.*

Proof. Let $u \in U$.

Let $u \to v_1$. By 1.3, u is comparable with both v_2 and v_1v_2. If $v_2 \to u$, then $u \to v_1v_2$ by (p5) and we get a contradiction by 1.1(3). Hence $u \to v_2$, and then $u \to v_1v_2$.

Now let $u \to v_1v_2$. By 1.3, u is comparable with both v_1 and v_2. We cannot have $v_1 \to u$ and $v_2 \to u$ at the same time, since then $v_1v_2 \to u$. Hence either $u \to v_1$ or $u \to v_2$. But then we have both $u \to v_1$ and $u \to v_2$ by the first part of the proof.

This proves that for any $u \in U$, $u \to v_1$ iff $u \to v_2$ iff $u \to v_1v_2$.

Let $u||v_1$. Then $uv_1 \to v_1$ implies $uv_1 \to v_2$ and $uv_1 \to v_1v_2$. We have $v_1v_2u = v_1uv_2v_1u = v_1uv_1u = v_1u$. Hence $u||v_1v_2$. We cannot have $u \to v_2$. If $v_2 \to u$, then $v_1v_2 \to v_2 \to u$ and $uv_1 \to v_2$ contradict (p5). Hence $u||v_2$. Similarly as for v_1, we get $v_1v_2u = v_2u$.

The rest is clear. □

1.5. **Proposition.** *Let $u_1, u_2 \in U$ and $v \in V$ be such that $u_1||u_2$ and $u_1 \to v \to u_2$. Then $v \to u_1u_2$ and there is no $w \in V$ with $u_2 \to w \to u_1$.*

Proof. If $v||u_1u_2$, then $u_1u_2 \to u_1 \to v$ contradicts 1.2. By (p5) we get $v \to u_1u_2$. Suppose there is an element $w \in V$ with $u_2 \to w \to u_1$. Then $w \to u_1u_2$, and $v \updownarrow w$ by 1.4. But then the elements u_1, u_2, v, w, u_1u_2 constitute a subalgebra isomorphic to $\mathbf{J}_3$, a contradiction. □

1.6. **Proposition.** *Let $u \in U$ and $v_1, v_2 \in V$ be such that $u||v_1$ and $u||v_2$. Then $uv_1 = uv_2$.*

Proof. Suppose $uv_1 \neq uv_2$. By 1.4, $v_1 \updownarrow v_2$. Without loss of generality, we can assume that $v_1 \to v_2$. By 1.3, $uv_1 \updownarrow v_2$. If $uv_1 \to v_2$ then $uv_2v_1 = uv_1v_2uv_1 = uv_1$, so that $uv_2||v_1$, a contradiction by 1.3. Hence $v_2 \to uv_1$. From $uv_2v_1 = v_2uv_1 = v_2v_1uv_2v_1 = v_1$ we get $v_1 \to uv_2$. If $uv_1||uv_2$, we get a contradiction by the second part of 1.5. Hence $uv_1 \updownarrow uv_2$. But then, by (p5), both $uv_1 \to uv_2$ and $uv_2 \to uv_1$, a contradiction. □

1.7. **Proposition.** *Let $u \in U$, $v \in V$, $u||v$. Then for every $w \in V$ either $uw = uv$ or else $w \to u$ and $w \to uv$.*

Proof. By 1.3 we cannot have $u \to w$. If $u||w$, then $uw = uv$ by 1.6. It remains to consider the case $w \to u$. By 1.4, $v \updownarrow w$. If $w \to v$, then clearly $w \to uv$. Finally, let $v \to w$. By 1.3 we have $uv \updownarrow w$, and hence $w \to uv$ by (p5). □

2. Incomparabilities in V

By a basic pair we will mean a pair a, b of elements of V such that either $a||b$ or $b = ad$ for some $d \in V$ with $d||a$ or $a = bd$ for some $d \in V$ with $d||b$. In this section we assume that there exists a basic pair a, b and a sequence $c_1, \dots, c_n$ of elements of V such that $ac_1 \dots c_n \not\equiv bc_1 \dots c_n$. Then let us consider one such sequence $a, b, c_1, \dots, c_n$ minimal in the sense that n is as small as possible and, among all such sequences of the same length, the number $Y = |\{i : ac_1 \dots c_{i-1}||c_i\}| + |\{i : bc_1 \dots c_{i-1}||c_i\}|$ is as small as possible. By 1.4, we have $n \geq 1$.

Two elements v, v' of V are said to be connected through basic pairs if there exists a finite sequence $v_0, \dots, v_k$ of elements of V such that $v_0 = v$, $v_k = v'$ and for each $j = 1, \dots, k$, v_{j-1}, v_j is a basic pair.

2.1. **Proposition.** *Let $i \in \{1, \dots, n\}$. Then $ac_1 \dots c_i \neq bc_1 \dots c_i$ and the elements $ac_1 \dots c_i$ and $bc_1 \dots c_i$ are not connected through basic pairs.*

Proof. Suppose the elements are connected through $v_0, \dots, v_k$. For each $j = 1, \dots, k$ we have $v_{j-1}c_{i+1} \dots c_n \equiv v_jc_{i+1} \dots c_n$ by the minimality of n. Hence, by the transitivity of $\equiv$, $ac_1 \dots c_n \equiv bc_1 \dots c_n$, a contradiction. □

2.2. **Proposition.** *$c_1 \updownarrow a$ and $c_1 \updownarrow b$.*

Proof. It is easy to see that if either $c_1||a$ or $c_1||b$, then (in every one of a small number of possible cases) ac_1 and bc_1 are connected through basic pairs, a contradiction with 2.1. □

2.3. **Proposition.** *If $b = ad$ for some $d||a$, then $a \to c_1 \to b$ and $c_1 \to d$.*

Proof. Suppose $c_1 \to a$. Due to 2.1 and 2.2, $b \to c_1$. But then $c_1 d = b$ and c_1, b is a basic pair, a contradiction. Hence $a \to c_1$. Then $c_1 \to b$ and, by (p3), $c_1 \to d$. □

2.4. **Proposition.** *If $a||b$ then either $a \to c_1 \to b$ and $c_1 \to ab$, or else $b \to c_1 \to a$ and $c_1 \to ab$.*

Proof. Clearly, either $a \to c_1 \to b$ or $b \to c_1 \to a$. By symmetry, it is sufficient to consider the first case. Then $ac_1 = a$ and $bc_1 = c_1$. If $c_1||ab$, then a, ab and ab, c_1 are basic pairs, a contradiction. Hence $c_1 \updownarrow ab$ and $c_1 \to ab$ by (p5). □

It follows from these lemmas that without loss of generality, we can assume that $a||b$, $a \to c_1 \to b$ and $c_1 \to ab$. So, we will go on under this assumption. We will assume that we have already proved for some index i the following: $a \to c_1 \to \dots \to c_i \to b$, $c_j \to b$ for all $j \leq i$, $c_j \to a$ for all $2 \leq j \leq i$, $c_k \to c_j$ for $1 \leq j < j+2 \leq k \leq i$, $c_j \to ab$ for all $j \leq i$, and $a \equiv c_1 \equiv \dots \equiv c_{i-1} \equiv b$. (This has been proved for $i = 1$.)

Put $c_0 = a$. Clearly, $\{ac_1 \dots c_j, bc_1 \dots c_j\} = \{c_{j-1}, c_j\}$ for $1 \leq j \leq i$.

2.5. **Proposition.** *$c_i \equiv a$. Consequently, $n > i$.*

Proof. Let $u \in U$. Let $a \to u$, so that also $b \to u$, $ab \to u$ and $c_j \to u$ for $j < i$. Suppose $u \to c_i$. Then all these elements constitute a subalgebra isomorphic to $\mathbf{M}_{i+2}$, a contradiction. So, $a \to u$ implies that either $c_i \to u$ or $u||c_i$.

Let $c_i \to u$. Suppose $u \to a$. Then all these elements together with e (with respect to $a \to c_1 \to \dots \to c_i \to u \to b$) constitute a subalgebra isomorphic to $\mathbf{M}_{i+3}$, a contradiction. So, $c_i \to u$ implies that either $a \to u$ or $a||u$.

If $u \to c_i$ then by 1.3 we cannot have $a||u$, so we get $a \to u$. If $u \to a$ then we cannot have $u||c_i$, so we get $u \to c_i$. So, $u \to a$ if and only if $u \to c_i$.

Let $u||c_i$. Then $uc_i \in U$ and $uc_i \to c_i$. Hence $uc_i \to a$. By 1.7 we get $ua = uc_i$. Quite similarly, if $u||a$ then $uc_i = ua$. The rest is clear. □

2.6. **Proposition.** *$c_{i+1} \updownarrow c_i$.*

Proof. Suppose $c_{i+1}||c_i$. If also $c_{i+1}||c_{i-1}$ then $c_{i-1}c_{i+1}, c_i c_{i+1}$ can be connected through basic pairs, a contradiction. If $c_{i+1} \to c_{i-1}$ then $c_{i-1}c_{i+1}$, $c_i c_{i+1}$ is a basic pair, a contradiction. Hence $c_{i-1} \to c_{i+1}$ and thus $c_{i-1} \to c_i c_{i+1}$. We have $\{c_{i-1}c_{i+1}, c_i c_{i+1}\} = \{c_{i-1}, c_i c_{i+1}\}$. But then c_{i+1} can be replaced with $c_i c_{i+1}$, a contradiction with the minimality of Y. □

2.7. **Proposition.** *$c_{i+1} \updownarrow c_{i-1}$.*

Proof. Suppose $c_{i+1}||c_{i-1}$. If $c_{i+1} \to c_i$ then $c_{i-1}c_{i+1}, c_ic_{i+1}$ is a basic pair, a contradiction. If $c_i \to c_{i+1}$ then $\{c_{i-1}c_{i+1}, c_ic_{i+1}\} = \{c_{i-1}c_{i+1}, c_i\}$, $c_{i-1}c_{i+1} \updownarrow c_i$, $c_i \to c_{i-1}c_{i+1}$ and c_{i+1} can be replaced with $c_{i_1}c_{i+1}$, a contradiction with the minimality of Y. □

2.8. **Proposition.** $c_i \to c_{i+1} \to c_{i-1}$.

Proof. Suppose, on the contrary, that $c_{i-1} \to c_{i+1} \to c_i$, so that $\{c_{i-1}c_{i+1}, c_ic_{i+1}\} = \{c_{i-1}, c_{i+1}\}$. Of course, $i > 1$.

Supppose there is an index j with $1 \leq j < i-1$ and $c_j \not\to c_{i+1}$, and let j be the largest index with that property. If $c_j||c_{i+1}$, then this is a basic pair and $\{c_jc_{j+1}, c_{i+1}c_{j+1}\} = \{c_j, c_{j+1}\}$, a contradiction with the minimality of n. Hence $c_{i+1} \to c_j$. By the minimality of n, $c_jc_{i+1}\dots c_n \equiv c_{j+1}c_{i+1}\dots c_n$, i.e., $c_{i+1}\dots c_n \equiv c_{j+1}c_{i+2}\dots c_n$. But also $c_{j+1}c_{i+2}\dots c_n \equiv c_{j+2}c_{i+2}\dots c_n \equiv \dots \equiv c_{i-1}c_{i+2}\dots c_n$ and hence $c_{i-1}c_{i+2}\dots c_n \equiv c_{i+1}c_{i+2}\dots c_n$, a contradiction. We have proved that $c_j \to c_{i+1}$ for all $1 \leq j \leq i-1$.

Suppose $a||c_{i+1}$. Then $ac_{i+2}\dots c_n \equiv c_{i+1}c_{i+2}\dots c_n$, but also $ac_{i+2}\dots c_n \equiv c_1c_{i+2}\dots c_n \equiv \dots c_{i-1}c_{i+2}\dots c_n$, so that $c_{i-1}c_{i+2}\dots c_n \equiv c_{i+1}c_{i+2}\dots c_n$, a contradiction.

Suppose $c_{i+1} \to a$. Then $ac_{i+1}c_{i+2}\dots c_n \equiv c_1c_{i+1}\dots c_n$, i.e., $c_{i+1}c_{i+2}\dots c_n \equiv c_1c_{i+2}\dots c_n$. But also $c_1c_{i+2}\dots c_n \equiv c_2c_{i+2}\dots c_n \equiv \dots \equiv c_{i-1}c_{i+2}\dots c_n$, so that $c_{i-1}c_{i+2}\dots c_n \equiv c_{i+1}c_{i+2}\dots c_n$, a contradiction.

Hence $a \to c_{i+1}$.

Suppose $b||c_{i+1}$. Then $c_{i+1}c_ic_{i+2}\dots c_n \equiv bc_ic_{i+2}\dots c_n$, i.e., $c_{i+1}c_{i+2}\dots c_n \equiv c_ic_{i+2}\dots c_n$. But also $c_{i-1}c_{i+2}\dots c_n \equiv c_ic_{i+2}\dots c_n$ and thus $c_{i-1}c_{i+2}\dots c_n \equiv c_{i+1}c_{i+2}\dots c_n$, a contradiction.

Suppose $c_{i+1} \to b$. Then $ac_{i+1}c_{i+2}\dots c_n \equiv bc_{i+1}c_{i+2}\dots c_n$, i.e., $ac_{i+2}\dots c_n \equiv c_{i+1}c_{i+2}\dots c_n$. But also $ac_{i+2}\dots c_n \equiv c_1c_{i+2}\dots c_n \equiv \dots \equiv c_{i-1}c_{i+2}\dots c_n$, so that $c_{i-1}c_{i+2}\dots c_n \equiv c_{i+1}c_{i+2}\dots c_n$, a contradiction.

Hence $b \to c_{i+1}$. Then also $ab \to c_{i+1}$. But then all these elements constitute a subalgebra isomorphic to $\mathbf{M}_{i+2}$, a contradiction. □

2.9. **Proposition.** $c_{i+1} \to c_j$ *for all* $1 \leq j \leq i-1$.

Proof. Suppose, on the contrary, that j is the largest index with $1 \leq j < i-1$ and $c_{i+1} \not\to c_j$. If $c_{i+1}||c_j$ then $c_{i+1}c_{i+2}\dots c_n \equiv c_jc_{i+2}\dots c_n \equiv c_{j+1}c_{i+2}\dots c_n \equiv \dots \equiv c_ic_{i+2}\dots c_n$, a contradiction. If $c_j \to c_{i+1}$ then $c_jc_{i+1}c_{i+2}\dots c_n \equiv c_{j+1}c_{i+1}c_{i+2}\dots c_n$, i.e., $c_jc_{i+2}\dots c_n \equiv c_{i+1}c_{i+2}\dots c_n$, but also $c_jc_{i+2}\dots c_n \equiv c_{j+1}c_{i+2}\dots c_n \equiv \dots \equiv c_ic_{i+2}\dots c_n$, so that $c_ic_{i+2}\dots c_n \equiv c_{i+1}c_{i+2}\dots c_n$, a contradiction. □

2.10. **Proposition.** $c_{i+1} \to a$.

Proof. . If $a||c_{i+1}$, then a contradiction can be obtained in the same way as in 2.9, with $c_j = c_0$. If $a \to c_{i+1}$ then $ac_{i+1}c_{i+2}\dots c_n \equiv c_1c_{i+1}c_{i+2}\dots c_n$, i.e., $ac_{i+2}\dots c_n \equiv c_{i+1}c_{i+2}\dots c_n$, but also $ac_{i+2}\dots c_n \equiv c_1c_{i+2}\dots c_n \equiv \dots \equiv c_ic_{i+2}\dots c_n$, so that $c_ic_{i+2}\dots c_n \equiv c_{i+1}c_{i+2}\dots c_n$, a contradiction. □

2.11. **Proposition.** $c_{i+1} \to b$ *and* $c_{i+1} \to ab$.

Proof. If $c_{i+1}||b$ then $c_{i+1}c_{i+2}\dots c_n \equiv bc_{i+2}\dots c_n \equiv ac_{i+2}\dots c_n \equiv c_1c_{i+2}\dots c_n \equiv \dots \equiv c_ic_{i+2}\dots c_n$, a contradiction. Suppose $b \to c_{i+1}$. Then $c_{i+1}||ab$, since otherwise $c_{i+1} \to ab$ and $b \to c_{i+1} \to a$ with c_1 and ab would give a subalgebra isomorphic to $\mathbf{J}_3$. Hence $c_{i+1}c_{i+2}\dots c_n \equiv (ab)c_{i+2}\dots c_n \equiv ac_{i+2}\dots c_n \equiv c_1c_{i+2}\dots c_n \equiv \dots \equiv c_ic_{i+2}\dots c_n$, a contradiction. Hence $c_{i+1} \to b$ and, consequently, $c_{i+1} \to ab$. □

The assumption taken at the beginning of this section turns out to be contradictory, as by 2.5 we get $n > i$ for all positive integers i. As a consequence, we get the following result.

2.12. **Theorem.** *Let A, B be two algebras in $\mathcal{T}^*$ such that B is an extension of A by an element e, and let $V = \{a \in A : a \to e\}$. The congruence of B generated by the pairs $(a,b) \in V^2$ such that $a||b$ is contained in $V^2 \cup \mathrm{id}_B$.*

3. More results

3.1. **Proposition.** *Let $u \in U$, $v \in V$ and $u||v$. Then there is no $a \in A$ with $u \to a \to uv$.*

Proof. Suppose there is. We have $a \to v$ by (p3), a contradiction with 1.2. □

3.2. **Proposition.** *Let $u_1, u_2 \in U$ and $v \in V$ be such that $u_1||u_2$ and $u_1 \to v \to u_2$. Then there is no $w \in V$ with $u_2 \to w$.*

Proof. Suppose there is. Since $u_1 \to v$, by 1.3 we cannot have $u_1||w$. By 1.5 we have $v \to u_1u_2$ and we cannot have $w \to u_1$. Hence $u_1 \to w$. Since $v \to u_2 \to w$, by 1.4 we cannot have $v||w$. If $w \to v$ then these elements constitute a subalgebra isomorphic to $\mathbf{M}_3$, a contradiction. Hence $v \to w$. But then these elements together with e (with $u_1 \to v \to e \to u_2$) constitute a subalgebra isomorphic to $\mathbf{M}_4$, a contradiction. □

3.3. **Proposition.** *Let $u \in U$, $v \in V$, $u||v$. Then for any $s \in A$, $s \to uv$ implies $s \to u$.*

Proof. Let $s \to uv$. Let us first consider the case $s \in V$. If $s||u$ then by 1.6 we have $us = uv$, a contradiction with $s \to uv$. If $u \to s$, we get a contradiction by 3.1. Hence $s \to u$.

Now consider the case $s \in U$. Again by 3.1, we cannot have $u \to s$. Suppose $s||u$. Since $s \to uv \to v$, by 1.2 we cannot have $s||v$. If $v \to s$ then $s \to u$ by (p3). So, let $s \to v$. Since $us \to s \to v$, by 1.2 we cannot have $us||v$. If $us \to v$ then $us \to uv$, a contradiction with (p5). Hence $v \to us$. But then $v \to u$ by (p3), a contradiction. □

3.4. **Proposition.** *Let $u \in U$, $v \in V$, $u||v$. Then for any $s \in A$, $u \to s$ implies $uv \to s$.*

Proof. Let $u \to s$. Then $s \in U$ by 1.3. By 3.1, $s \not\to uv$. So, suppose $s||uv$. By 3.1, we cannot have $u \to uvs$. Hence, by (p5), $u||uvs$ and $uv \to uvsu$.

By (p1) we get $v||uvsu$ and $v \cdot uvsu = uv$. But $uvsu \to uvs \to uv$, a contradiction by 3.1. □

3.5. **Proposition.** *Let $u \in U$, $v \in V$, $u||v$. Then there are no elements $r, s \in U$ with $u \to r \to s \to uv$.*

Proof. Suppose there are. By 3.3 and 3.4, $s \to u$ and $uv \to r$.

Suppose $s \to v$. Then, by 1.2, we cannot have $r||v$. Again by 1.2, we cannot have $r \to v$. Hence $v \to r$. But then these elements together with e (with respect to $v \to e \to s \to u$) constitute a subalgebra isomorphic to $\mathbf{M}_4$, a contradiction.

Since $s \to uv \to v$, by 1.2 we cannot have $s||v$. It follows that $v \to s$.

By 1.2 we cannot have $r \to v$. If $v \to r$ then these elements, with respect to $v \to s \to u$, constitute a subalgebra isomorphic to $\mathbf{M}_3$, a contradiction. Hence $v||r$. We have $vru = vurvu = uvuv = uv$. Consequently, the elements r, s, u, vr, uv (with respect to $vr \to s \to u$) constitute a subalgebra isomorphic to $\mathbf{M}_3$, a contradiction. □

3.6. **Proposition.** *Let $a, b, p \in U$ and $v \in V$ be such that $a||v$, $b \to a$, $p \to a$ and $av = bv$. Then $bpv = pv$.*

Proof. Let $p \to v$. Then $p \to av = bv \to b$, so $p \to b$ by 3.3. Hence $bp = p$ and $bpv = pv$.

Let $v \to p$. Then $bpv = pbv = pvbpv = vbpv = vapv = avpv = apvp = pvp = pv$.

It remains to consider the case $p||v$. Since $pv \to p \to a$, by 3.4 we have $pv \to a$. Hence $pv \to av$. We have $avp = apvap = pvap = pvp = pv$. By three-variable equations, $bpv \cdot pv = bvpv = avpv = pvv = pv$, so that $pv \to bpv$. We have $bpvp = bvpv = pv$.

If either $bp||v$ or $bp \to v$ then $bpv \to p$, $bpvp = bpv$, so $bpv = pv$ and we are through. So, the case $v \to bp$ remains. Then $v = bpv = bvpbv = avpbv = pvbv = pbvb = vb$, a contradiction. □

3.7. **Proposition.** *Let $u \in U$, $v \in V$, $u||v$; let $a \in U$. Then $uv \cdot ua = uva$ and $uvaw = uaw$ for all $w \in V$.*

Proof. Since $uva \to uv$, we have $uva \to u$ by 3.3. Hence $uv \cdot ua = uv \cdot ua \cdot u = a \cdot u \cdot uv \cdot u = a \cdot uv \cdot u \cdot uv = uvau \cdot uv = uva \cdot uv = uva$. In order to prove the rest, it is sufficient to assume that $a \to u$. By 1.7 we have either $uw = uv$ or $uvw = uw = w$, so $uvw = uw$ in any case. Hence, by 3.6, it is sufficient to consider the case $u \updownarrow w$. By 1.7 we have $w \to u$ and $w \to uv$.

If $w \to a$ then $w \to uva$ and $uvaw = w = aw$.

Let $a \to w$. Then $a \updownarrow v$. If $a \to v$ then $uva = uavua = a$ and we are through. So, let $v \to a$. Then $v \to a \to u$ gives $v \to uva$ by (p5). We have $uv \to v \to a$, $a \to w \to uv$ and (obviously) $uv||a$, a contradiction by 1.5.

It remains to consider the case $a||w$. Then $aw \to u$ by 3.4. Since $aw \to w$, by 1.3 we cannot have $aw||v$. If $aw \to v$ then $aw \to uv$, hence $aw \to uva$, and $aw \to uva \to a$ implies $uvaw = aw$ by (p1). So, let $v \to aw$. We have

$uvaw = uvwa(uv)w = (aw \cdot uv)w$. By the previous part of the proof (the case $a \to w$) we have $(uv \cdot aw)w = aww = aw$. Hence $uvaw = aw$. □

3.8. **Proposition.** *Let $u \in U$, $v \in V$, $u||v$; let $a \in U$ be such that $a \to u$ and $a||uv$. Then there is no element $b \in U$ with $a \to b \to uva$.*

Proof. Suppose there is. We have $uvav = uava = av$. So, if $uva \to v$ then $av = uva$, a contradiction with $a \to b \to uva$ by 3.1. Since $uva \to uv \to v$, we cannot have $uva||v$. Hence $v \to uva$. From $uvav = av$ we get $v \to a$. By (p3), $b \to uv$. Since $b \to uv \to v$, we cannot have $b||v$. Now either $b \to v$ or $v \to b$, and in each case the elements uv, v, a, b, uva constitute a subalgebra isomorphic to $\mathbf{J}_3$, a contradiction. □

3.9. **Proposition.** *Let $u_1, u_2 \in U$, $v, w \in V$, $u_1||u_2$, $u_1 \to v \to u_2$ and $u_2||w$. Then one of the following two cases takes place:*

(1) *$u_1u_2 = u_2w$, $v \to u_1u_2$, $u_1 \updownarrow w$, $v \updownarrow w$;*
(2) *$v \to w \to u_1$, $v \to u_1u_2$, $v \to u_2w \to u_1$, $u_1u_2w = u_2w$.*

Proof. We have $u_1 \updownarrow w$ by 1.3 and $v \updownarrow w$ by 1.4. Let $u_1u_2 \neq u_2w$. Since $v \to u_2$, we have $v \to u_2w$ by 1.7. Since $u_1 \to v \to u_2w \to w$, we have $u_1 \updownarrow u_2w$ by 3.2. If $u_1 \to u_2w$ then $u_1 \to u_2$ by 3.3, a contradiction. Hence $u_2w \to u_1$. Since also $u_2w \to u_2$, we get $u_2w \to u_1u_2$. Since $u_2w \to u_1u_2 \to u_2$, by (p2) we get $u_1u_2w = u_2w$. If $u_1 \to w$ then $u_1u_2 = u_2w$ by (p2), a contradiction. Since $u_1 \updownarrow w$, we get $w \to u_1$. It remains to prove $v \to w$. We have $v \updownarrow w$, and if $w \to v$ then the elements w, v, u_1u_2, u_2w, u_1 (with respect to $w \to v \to u_1u_2$) constitute a subalgebra isomorphic to $\mathbf{M}_3$, a contradiction. □

3.10. **Proposition.** *Let $u_1, u_2 \in U$, $v \in V$, $u_1||u_2$, $u_1 \to v \to u_2$. Then for every $w \in V$ one of the following cases takes place:*

(1) *$u_2||w$, $u_1u_2 = u_2w$, $v \to u_1u_2$, $u_1 \updownarrow w$, $v \updownarrow w$;*
(2) *$u_2||w$, $v \to w \to u_1$, $v \to u_1u_2$, $v \to u_2w \to u_1$, $u_1u_2w = u_2w$;*
(3) *$w \to u_2$, $w \to u_1u_2$, $v \to u_1u_2$, $w \updownarrow u_1$, and if $w \to u_1$ then $w \updownarrow v$.*

Proof. By 3.2 and 3.9, it remains to consider the case $w \to u_2$. According to 1.3 we have $w \updownarrow u_1$, and according to 1.4 if $w \to u_1$ then $w \updownarrow v$. By 1.5, $v \to u_1u_2$.

Suppose $w||u_1u_2$. By 3.4 we have $u_1u_2w \to u_1$ and $u_1u_2w \to u_2$. If $u_1 \to w$ then $u_1 \to w \to u_2$ implies $u_1 \to u_1u_2w$ by (p5), a contradiction. Hence $w \to u_1$. But then $w \to u_1u_2$, a contradiction.

Hence $w \updownarrow u_1u_2$. It follows that if $u_1 \to w$ then $w \to u_1u_2$. If $w \to u_1$, then $w \to u_1u_2$ is clear. So, $w \to u_1u_2$ in all cases. □

3.11. **Proposition.** *Let $u_1, u_2 \in U$, $v \in V$, $u_1||u_2$, $u_1 \to v \to u_2$. Then there is no element $u \in A$ with $u_2 \to u \to u_1u_2$, and there is no element $u \in A$ with $u_2 \to u \to u_1$.*

Proof. In each case, we would have $u \in U$ according to 3.2. By 1.5 we have $v \to u_1u_2$. Suppose $u_2 \to u \to u_2u_2$. By (p3), $u \to u_1$. Since $u \to$

$u_1 \to v$, by 1.2 we cannot have $u||v$. But then, the elements u_1, u, u_2, u_1u_2, v constitute a subalgebra isomorphic to $\mathbf{J}_3$, a contradiction.

Now suppose $u_2 \to u \to u_1$. Then $u_2 \to u_1u_2u \to u_1u_2$, which has been proved to be impossible. □

3.12. **Proposition.** *Let $u \in U$, $v \in V$, $u||v$ and $c_i \in U$ $(i = 1, \dots, n)$ be elements with $c_n \to c_{n-1} \to \dots c_1 \to u$. Then $uvc_1 \dots c_nv = c_nv$.*

Proof. The quasiequation $z_n \to z_{n-1} \to \dots \to z_1 \to x \implies xyz_1 \dots z_ny = z_ny$ is satisfied in all tournaments and is equivalent to an equation, so it is satisfied in A. □

3.13. **Proposition.** *Let n be the least number for which there exist elements $u \in U$, $v \in V$, $w \in V$ and $c_i \in U$ $(i = 1, \dots, n)$ such that $u||v$, $c_n \to c_{n-1} \to \dots \to c_1 \to u$ and $uvc_1 \to c_nw \neq c_nw$. Then*

(1) *$v \to c_i$ and $v \to uvc_1 \dots c_i$ for all $i \geq 1$.*
(2) *$w \to c_{n-1}$, $w \to uvc_1 \dots c_{n_1}$ and $w \to uvc_1 \dots c_n$.*
(3) *It is sufficient to consider only the case $c_n \to w$.*

Proof. By 3.7 we have $n \geq 2$. Suppose that for some i, $v \not\to uvc_1 \dots c_i$. By 3.12, $uvc_1 \dots c_iv = c_iv$. If $c_iv = c_i$ then $uvc_1 \dots c_iv = c_i$, so that $c_i \to uvc_1 \dots c_i$ and hence $uvc_1 \dots c_i = c_i$, a contradiction. Hence $c_i||v$. By the minimality of n, $c_nw = c_ivc_{i+1} \dots c_nw = uvc_1 \dots c_ivc_{i+1} \dots c_nw$. Hence $uvc_1 \dots c_iv \neq uvc_1 \dots c_i$. Using $uvc_1 \dots c_n \to uvc_1 \dots c_{n-1} \to \dots \to uvc_1 \dots c_i$, by the minimality of n we have

$$\begin{aligned} uvc_1 \dots c_nw &= uvc_1 \dots c_iv(uvc_1 \dots c_{i+1}) \dots (uvc_1 \dots c_n)w \\ &= vc_i(uvc_1 \dots c_{i+1}) \dots (uvc_1 \dots c_n)w. \end{aligned}$$

But this last expression equals $vc_ic_{i+1} \dots c_nw$, since the quasiequation

$$z_n \to \dots z_1 \to x \implies uz_i \dots z_n = yz_i(xyz_1 \dots z_{i+1}) \dots (xyz_1 \dots z_n)$$

is satisfied in all tournaments and is equivalent to an equation. We get $uvc_1 \dots c_nw = vc_ic_{i+1} \dots c_nw = c_nw$, a contradiction.

Hence $v \to uvc_1 \dots c_i$ for all i. From this we get $v \to c_i$ by (p3).

We have $c_{n-1}w = uvc_1 \dots c_{n-1}w$ by the minimality of n. If $w||c_{n-1}$ then $c_nw = uvc_1 \dots c_{n-1}c_nw$ by 3.6, a contradiction. Hence $w \to c_{n-1}$. Consequently, $w \to uvc_1 \dots c_{n-1}$.

Suppose $w \not\to uvc_1 \dots c_n$. Then $uvc_1 \dots c_nw \to uvc_1 \dots c_n \to c_n$ implies $uvc_1 \dots c_nw \to c_n$; hence $uvc_1 \dots c_nw \to wc_n$. We get

$$uvc_1 \dots c_{n-1}wc_n = (uvc_1 \dots c_{n-1}w \cdot uvc_1 \dots c_n)(uvc_1 \dots c_{n-1}w \cdot wc_n),$$

i.e.,

$$wc_n = uvc_1 \dots c_nw \cdot wc_n = uvc_1 \dots c_n,$$

a contradiction.

Hence $w \to uvc_1 \dots c_n$. Then $w \not\to c_n$ and $wc_{\to}c_{n-1}$. The quasiequation

$$\begin{aligned} y \to z_n \to \dots \to z_1 \to x \implies & xyz_1 \dots z_{n-1} \cdot uz_{n-1}z_nz_{n-1} = \\ & xyz_1 \dots z_n \cdot uz_{n-1}z_nz_{n-1} \end{aligned}$$

is satisfied in all tournaments and is equivalent to an equation; we get $uvc_1 \dots c_{n-1} \cdot wc_n = uvc_1 \dots c_n \cdot wc_n$. From this it follows that if c_n is replaced with wc_n, all the above conditions are satisfied and, moreover, $c_n \to w$. □

3.14. **Proposition.** *Let $u \in U$, $v \in V$, $u||v$, $c_1, c_2 \in U$, $c_2 \to c_1 \to u$. Then $uvc_1c_2w = c_2w$ for all $w \in V$.*

Proof. Suppose $uvc_1c_2w \neq c_2w$. By 3.13 we have $v \to c_1$, $v \to c_2$, $v \to uvc_1$, $v \to uvc_1c_2$, $w \to c_1$, $w \to uvc_1$, $w \to uvc_1c_2$ and it is sufficient to consider the case $c_2 \to w$. Since $uv \to v \to uvc_1c_2$ and (by (p5)) $uvc_1c_2 \to uv \cdot uvc_1c_2 \cdot uvc_1 \to uv \cdot uvc_1c_2$, by 3.11 we have $uv \updownarrow uvc_1c_2$. Since $uv \to v \to c_2$ and $c_2 \to w$, by 3.2 we have $uv \updownarrow c_2$. If $c_2 \to uv$ then $c_2 \to uvc_1$, so that $uvc_1c_2 = c_2$, a contradiction. Hence $uv \to c_2$. Then $uv \to uvc_1c_2$. But $c_2||uvc_1$, so that $c_2 \to w \to uvc_1$ and $uvc_1 \to uv \to uvc_1c_2$ give a contradiction by 3.11. □

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