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ON INJECTIVE HULLS

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ABSTRACT. It follows immediately from the proof of the existence of the injective hull $E(M)$ of a module M (see e.g. [1]) that an upper bound for the cardinality of $E(M)$ is $|M|^{|R|}$. On the other hand, over left noetherian rings the injective hull $E(M)$ has the same size as M whenever $|M| \geq \tilde{\mu}_0 = \max(|R|, \aleph_0)$ (see [5]). Denoting $\tilde{\mu} = \sup \{\mu_I \mid I \leq R\}$, where μ_I denotes the smallest cardinality of a set of generators of the left ideal I and I ranges through all the left ideals of the ring R, we put $\mu = \tilde{\mu}$ and $\mu_0 = \tilde{\mu}_0$ if $\tilde{\mu}$ is regular and $\mu_I < \tilde{\mu}$ for any left ideal I of R and we put $\mu = \tilde{\mu}^+$ and $\mu_0 = \tilde{\mu}_0^+$ in the opposite case. The purpose of this note is to improve the above estimation by showing that for a module M of the size $|M| = \lambda \geq \mu_0^{<\mu}$ the cardinality of $E(M)$ is at most $\lambda^{<\mu}$.

In what follows, R stands for an associative ring with the identity element and by the word "module" we shall always mean an object of the category R-mod of all unital left R-modules. If M is an arbitrary module, then $E(M)$ denotes the *injective hull* of M and I_x will always denote the *annihilator left ideal* $(M : x) = \{r \in R \mid rx \in M\}$ of the element $x \in E(M)$ over M. For other notations and terminology from the theory of rings and modules we refer to [1].

For any set X the symbol $|X|$ will denote the *cardinality* of X. Similarly, for an ordinal α the cardinality of the set of all ordinals less than α is denoted by $|\alpha|$. Recall, that for the cardinals λ and μ the *weak power* $\lambda^{<\mu}$ is defined as the supremum of the set $\{\lambda^\alpha \mid \alpha < \mu\}$. For the details concerning the set theory, ordinals and the cardinal arithmetic see e.g. [9] or [2].

Let $M \in R$-mod be an arbitrary module of infinite cardinality. If D is the divisible hull of the additive group of the module M, then $|D| = |M|$ and it is well-known that M can be embedded into the injective R-module $\mathrm{Hom}_Z(R, D)$ (see e.g. [1]). From this we immediately get the inequalities $|M| \leq |E(M)| \leq |M|^{|R|}$. The purpose of this note is to improve this estimation

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by presenting the Theorem below. It should be also noted that over left noetherian rings the size of the injective hull $E(M)$ of the module M is the same as that of M whenever $|M| \geq \max(|R|, \aleph_0)$ (see [5]).

Throughout this note we shall put $\tilde{\mu}_0 = \max(|R|, \aleph_0)$ and $\tilde{\mu} = \sup\{\mu_I \,|\, I \leq R\}$, where μ_I denotes the smallest cardinality of a set of generators of the left ideal I and I ranges through all the left ideals of the ring R. Now we shall distinguish two cases. First, if $\tilde{\mu}$ is regular and $\mu_I < \tilde{\mu}$ for any left ideal I of R, then we put $\mu = \tilde{\mu}$ and $\mu_0 = \tilde{\mu}_0$. In the opposite case, i.e. if either $\tilde{\mu}$ is singular and $\mu_I < \tilde{\mu}$ for each $I \leq R$, or if $\tilde{\mu} = \max\{\mu_I \,|\, I \leq R\}$, then we put $\mu = \tilde{\mu}^+$ and $\mu_0 = \tilde{\mu}_0^+$. Note, that the trivial inequality $\tilde{\mu} \leq \tilde{\mu}_0$ immediately yields in both cases the inequalities $\mu \leq \mu_0 \leq \mu_0^{<\mu}$.

Now we are ready to prove our main result. The indirect proof is based on the construction of a strongly increasing chain of left ideals of R of the length μ, which trivially contradicts the definition of the cardinal μ.

Theorem 1. *If $M \in R$-mod is an arbitrary module such that $|M| = \lambda \geq \mu_0^{<\mu}$, then $\lambda \leq |E(M)| \leq \lambda^{<\mu}$.*

Proof. Proving indirectly, let us assume that $\lambda^{<\mu} < |E(M)| = \kappa = |E(M) \setminus M|$ for some module M with $|M| = \lambda \geq \mu_0^{<\mu}$. On the set $M_0 = E(M) \setminus M$ we define the equivalence relation $\sim_0$ in such a way that we put $x \sim_0 y$ if and only if $(M : x) = I_x = I_y = (M : y)$ and for each $r \in I_x$ the equality $rx = ry$ holds. Since it suffices to assume this equality for a minimal set of generators of the ideal I_x only, it is easy to see that there is at most $\tilde{\mu}_0^{<\mu}\lambda^{<\mu} = \lambda^{<\mu}$ equivalence classes under $\sim_0$. We denote by $P(\sim_0)$ the partition of the set M_0 corresponding to the equivalence relation $\sim_0$. Now for each $T \in P(\sim_0)$ we put $I_{0,T} = I_x$ for an (arbitrary) element $x \in T$ and we shall continue by the transfinite induction.

Assume, that for some ordinal $\alpha < \mu$, the sets M_β, the equivalence relations $\sim_\beta$ on M_β and the left ideals $I_{\beta,T}$, $T \in P(\sim_\beta)$, $P(\sim_\beta)$ being the partition of the set M_β corresponding to the equivalence relation $\sim_\beta$, have been already defined for each ordinal $\beta < \alpha$ in such a way that the following four conditions are satisfied:

(a) $M_\beta \subseteq M_\gamma$, $|M_0 \setminus M_\beta| \leq \lambda^{<\mu}$ and $|M_\beta| = \kappa$ for all $\gamma \leq \beta < \alpha$;
(b) for each $\beta < \alpha$ the partition $P(\sim_\beta)$ contains at most $\lambda^{<\mu}$ equivalence classes;
(c) the equivalence relation $\sim_\beta$ is a refinement of the relation $\sim_\gamma$ on the set M_β whenever $\gamma \leq \beta < \alpha$;
(d) for all $\gamma < \beta < \alpha$ it is $I_{\gamma,T} \subset I_{\beta,S}$ whenever $T \in P(\sim_\gamma)$, $S \in P(\sim_\beta)$ and $S \subseteq T$.

We start with the successor ordinal case. So, let $\alpha = \beta + 1$ and let $T \in P(\sim_\beta)$ be an arbitrary class. We fix an element $x_{\alpha,T} \in T$, we put $\overline{T} = \{x \in$

$T \mid x - x_{\alpha,T} \notin M\}$ and we finally set $M_\alpha = \cup_{T \in P(\sim_\beta)} \overline{T}$. Since $|T \setminus \overline{T}| \leq |M| = \lambda$ and there is at most $\lambda^{<\mu}$ equivalence classes under $\sim_\beta$, the obvious equality $M_0 \setminus M_\alpha = (M_0 \setminus M_\beta) \cup \bigcup_{T \in P(\sim_\beta)} (T \setminus \overline{T})$ yields that $|M_0 \setminus M_\alpha| \leq \lambda^{<\mu}$ and consequently the size of M_α is still κ. Further, we define the refinement $\sim_\alpha$ of the equivalence relation $\sim_\beta$ on the set M_α by setting $x \sim_\alpha y$ if and only if $x, y \in \overline{T}$ for some $T \in P(\sim_\beta)$, $I_{x - x_{\alpha,T}} = I_{y - x_{\alpha,T}}$ and $r(x - x_{\alpha,T}) = r(y - x_{\alpha,T})$ for each $r \in I_{x - x_{\alpha,T}}$. Finally we set $I_{\alpha,S} = I_{x - x_{\alpha,T}}$ whenever $S \in P(\sim_\alpha)$, $T \in P(\sim_\beta)$, $S \subseteq T$ and $x \in S$ is an arbitrary element. Clearly, every $T \in P(\sim_\beta)$ contains at most $\lambda^{<\mu}$ equivalence classes under $\sim_\alpha$ and consequently the condition (b) is satisfied. Now it clearly remains to verify the condition (d), we postpone it after the limit ordinal case.

So, we proceed to the limit ordinal case. In this part of the proof we shall see that the regularity of the cardinal μ is necessary for the estimation of the upper bound of equivalence classes. Setting $M_\alpha = \cap_{\beta < \alpha} M_\beta$ we have $M_0 \setminus M_\alpha = \cup_{\beta < \alpha} (M_0 \setminus M_\beta)$ and so $|M_0 \setminus M_\alpha| \leq |\alpha| \lambda^{<\mu} \leq \mu \lambda^{<\mu} = \lambda^{<\mu}$ and $|M_\alpha| = \kappa$, as desired. So, if $\sim_\alpha = \cap_{\beta < \alpha} \sim_\beta$ in the obvious sense, then the condition (c) is satisfied. Moreover, each equivalence class $S \in P(\sim_\alpha)$ is the intersection of the equivalence classes of $P(\sim_\beta)$, $\beta < \alpha$, and we are going to verify the condition (b). Assume first, that $\mu = \tilde{\mu}$ is a regular cardinal and that $\mu_I < \mu$ for each left ideal I of R. If $\emptyset \neq S \in P(\sim_\alpha)$ is arbitrary, then $S = \cap_{\beta < \alpha} S_\beta$, where $S_\beta \in P(\sim_\beta)$ for each $\beta < \alpha$. To every β non-limit it is associated a left ideal $I_\beta = I_{\beta, S_\beta}$ and there is at most λ^{μ_β}, $\mu_\beta = |\mu_{I_\beta}|$, equivalence classes of $\sim_\beta$ corresponding to the ideal I_β. Since there is at most $\tilde{\mu}_0^{<\mu}$ left ideals of R, there is at most $(\tilde{\mu}_0^{<\mu})^{|\alpha|}$ α-sequences of left ideals of R. To any such an α-sequence it corresponds at most $\prod_{\beta < \alpha} \lambda^{\mu_\beta}$ equivalence classes under $\sim_\alpha$. Now $\alpha < \mu$ and the regularity of μ yields that $\sup \{\mu_\beta \mid \beta < \alpha\} = \varrho < \mu$ and so $\prod_{\beta < \alpha} \lambda^{\mu_\beta} \leq \lambda^{<\mu}$. Summarizing we see that there is at most $(\mu_0^{<\mu})^{|\alpha|} \lambda^{<\mu} \leq \lambda^{|\alpha|} \lambda^{<\mu} = \lambda^{<\mu}$ equivalence classes under $\sim_\alpha$. Assume now that $\mu = \tilde{\mu}^+$. In this case there is at most $\tilde{\mu}_0^{\tilde{\mu}}$ left ideals of R and for each left ideal there is at most $\lambda^{\tilde{\mu}}$ equivalence classes in the above sense. Summarizing we see that there is at most $(\tilde{\mu}_0^{\tilde{\mu}} \lambda^{\tilde{\mu}})^{|\alpha|} = (\lambda^{\tilde{\mu}})^{|\alpha|} \leq \lambda^{<\mu}$ equivalence classes under $\sim_\alpha$. Finally, for each $\emptyset \neq S \in P(\sim_\alpha)$ we put $I_{\alpha,S} = \cup_{\beta < \alpha} I_{\beta, S_\beta}$ whenever $S_\beta \in P(\sim_\beta)$, $\beta < \alpha$, are such that $S = \cap_{\beta < \alpha} S_\beta$.

We proceed now to the verification of the condition (d). The main step here is in the case of non-limit ordinals $\alpha = \beta + 1$ and the methods are different in the case of β non-limit and β limit. Since the case of α limit is trivial, we may assume that $\alpha = \beta + 1$ is a non-limit ordinal and we shall distinguish two cases. Assume first, that $\beta = \gamma + 1$ is non-limit and let $S \in P(\sim_\alpha)$, $T \in P(\sim_\beta)$ and $U \in P(\sim_\gamma)$ be such that $S \subseteq T \subseteq U$. Recall, that $I_{\beta,T} = (M : (x - x_{\beta,U}))$ for all $x \in T$ and $I_{\alpha,S} = (M : (x - x_{\alpha,T}))$ for all $x \in S$. Especially, for each $r \in I_{\beta,T}$ and each $x \in S$ we have $r(x - x_{\alpha,T}) = r(x - x_{\beta,U}) - r(x_{\alpha,T} - x_{\beta,U}) = 0$, which means that $I_{\beta,T} \subseteq I_{\alpha,S}$. Moreover, the essentiality of M in $E(M)$ yields the existence of an element $r \in I_{\alpha,S}$ with $0 \neq r(x - x_{\alpha,T}) \in M$, which proves that the inclusion $I_{\beta,T} \subset I_{\alpha,S}$ is proper. So, let us assume now, that β is a limit ordinal. If $S \in P(\sim_\alpha)$ and $T \in P(\sim_\beta)$ are such that $S \subseteq T$, then $I_{\beta,T} = \cup_{\gamma<\beta} I_{\gamma,T_\gamma}$, where $T = \cap_{\gamma<\beta} T_\gamma$. For an arbitrary element $r \in I_{\beta,T}$ we have $r \in I_{\gamma,T_\gamma}$ for some $\gamma = \delta + 1 < \beta$ non-limit and so for all $x, y \in T$ we have $r(x - y) = r(x - x_{\gamma,T_\delta}) - r(y - x_{\gamma,T_\delta}) = 0$. This especially means that $I_{\beta,T}(x - x_{\alpha,T}) = 0$, hence $I_{\beta,T} \subseteq I_{\alpha,S}$ and the rest runs on the same lines as that in the case of β non-limit.

To finish the proof it suffices now to note that after μ steps we still obtain the set $M_\mu = \cap_{\alpha<\mu} M_\alpha$ of the size κ owing to the fact that $M_0 \setminus M_\mu = \cup_{\alpha<\mu}(M_0 \setminus M_\alpha)$ and consequently $|M_0 \setminus M_\mu| \leq \mu\lambda^{<\mu} \leq \lambda\lambda^{<\mu} = \lambda^{<\mu}$ in view of the simple inequalities $\mu \leq \mu_0 \leq \mu_0^{<\mu} \leq \lambda$ mentioned in the introduction. This yields the existence of a non-empty equivalence class $S \in P(\sim_\mu)$, where $\sim_\mu = \cap_{\alpha<\mu} \sim_\alpha$ in the natural sense. Thus for $S = \cap_{\alpha<\mu} S_\alpha$ the condition (d) gives the strongly increasing union $I_{\mu,S} = \cup_{\alpha<\mu} I_{\alpha,S_\alpha}$ of left ideals of R of the length μ. However, the left ideal $I = I_{\mu,S}$ has by the hypothesis a set of generators of the cardinality $\mu_I < \mu$. The regularity of μ yields the existence of an ordinal $\beta < \mu$ such that all these generators lie in I_{β,S_β}. Thus $I = I_{\beta,S_\beta} \subset I_{\beta+1,S_{\beta+1}} \subseteq I$ gives a contradiction finishing the proof of Theorem. □

Remark 2. Recall, that a *hereditary torsion theory* $\tau = (T_\tau, F_\tau)$ for the category R-mod consists of two abstract (closed under isomorphic copies) classes T_τ and F_τ, the *τ-torsion class* and the *τ-torsionfree class*, respectively, such that $\mathrm{Hom}(T, F) = 0$ whenever $T \in T_\tau$ and $F \in F_\tau$, the class T_τ is closed under submodules, factor-modules, extensions and arbitrary direct sums, the class F_τ is closed under submodules, extensions and arbitrary direct products and for each module M there exists an exact sequence $0 \to T \to M \to F \to 0$ such that $T \in T_\tau$ and $F \in F_\tau$. For a module M, the *singular submodule* $Z(M)$ consists of all elements $a \in M$, *the annihilator left ideal* $(0 : a) = \{r \in R \mid ra = 0\}$ of which is essential in R. *Goldie's torsion theory* for the category R-mod is the hereditary torsion theory $\sigma = (T_\sigma, F_\sigma)$, where $T_\sigma = \{M \in R\text{-mod} \mid Z(M/Z(M)) = M/Z(M)\}$ and $F_\sigma = \{M \in R\text{-mod} \mid Z(M) = 0\}$. Now it should be noted, that if the module M is torsionfree in Goldie's torsion theory (non-singular in other words), then the proof of the above Theorem is rather trivial. In this case for $x \sim_0 y$ we have $I_x(x - y) = 0$ and so $x = y$ in view of the fact that the left ideal I_x is essential in R and that the module M is torsionfree in the sense of Goldie. Since there is at most $\lambda^{<\mu}$ equivalence classes under $\sim_0$, the injective envelope $E(M)$ of M has at most $\lambda^{<\mu}$ elements.

Corollary 3. (See [5]). *If M is an arbitrary module over a left noetherian ring R such that $|M| = \lambda \geq \tilde{\mu}_0$, then $|E(M)| = |M| = \lambda$.*

Proof. Since every left ideal of a left noetherian ring R is finitely generated, it is obvious that $\tilde{\mu} = \mu = \aleph_0$, $\mu_0^{<\mu} = \mu_0$ and $\lambda^{<\mu} = \lambda$ and Theorem 1 applies. $\square$

Corollary 4. (GCH). *If M is an arbitrary module with $|M| = \lambda \geq 2^{\mu_0^+}$, where $\mu_0 = max\ (|R|, \aleph_0)$, then $|E(M)| = |M| = \lambda$ whenever λ is a regular cardinal.*

Proof. Under the generalized continuum hypothesis the regularity of λ yields the equality $\lambda^{<\mu} = \lambda$ and the above Theorem 1 applies. $\square$

References

[1] F.W. Anderson, K.R. Fuller, *Rings and Categories of Modules*, Graduate Texts in Mathematics **13**, Springer-Verlag (1974).

[2] B. Balcar, P. Štěpánek, *Teorie množin*, Academia Praha (1986).

[3] L. Bican, *A note on pure subgroups* Contributions to General Algebra **12**, Proceedings of the Vienna Conference, June 3-6, 1999, Verlag Johannes Heyn, Klagenfurt 2000, (2000), 105–107.

[4] L. Bican, *Pure subgroups*, Math. Bohemica **126** (2001), 649–652.

[5] L. Bican, *Relative purity over noetherian rings*, to appear.

[6] L. Bican, T. Kepka, P. Němec, *Rings, Modules, and Preradicals*, Marcel Dekker, New York (1982).

[7] B. Eckmann, A. Schopf, *Über injektive Moduln*, Archiv der Math. **4(2)** (1953), 75–78.

[8] J. Golan, *Torsion Theories*, Pitman Monographs and Surveys in Pure and Applied Matematics, **29**, Longman Scientific and Technical (1986).

[9] T. Jech, *Set theory*, Academic Press New York (1978).

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