J-Noetherian Bezout domain which are not of stable range 1. A Bezout ring of stable range 2 which has square stable range 1

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All rings considered will be commutative and have identity.

The notion of a stable range was useful in modern research on theory of diagonalization of matrices.

Recall that a ring $R$ is a **ring of stable range 1** if for any $a, b \in R$ such that $aR + bR = R$ we have $(a + bt)R = R$ for some $t \in R$. 
Studying an elementary divisor ring W. McGovern has introduced the concept of ring of almost stable range 1 as a ring whose proper homomorphic images all have stable range 1 [McGovern,2007].

By [McGovern,2007] a ring of stable range 1 is a ring of almost stable range 1. At the same time not every element of stable range 1 is an element of almost stable range 1.

Recall that element $a$ is an element of stable range 1 if for any $b \in R$ such that $aR + bR = R$ we have $a + bt$ is an invertible element.

An element $a \in R$ is of almost stable range 1 if $R/aR$ is a ring of stable range 1.

If $R = \mathbb{Z} \times \mathbb{Z}$, then $e = (1, 0)$ is element of stable range 1, but $R/eR \cong \mathbb{Z}$ is not a ring of stable range 1.
The problem of finding the element of almost stable range 1 in rings which are not rings of stable range 1 is required, in accordance with the above and considerations in [Zabavsky, 2017]. In this paper on the basis of theory comaximal factorization we prove that in any $J$-Noetherian Bezout ring which are not of stable range 1 there exist a nonunit element of almost stable range 1.

Unique factorization domains are, of course, integral domains in which every nonzero nonunit element has a unique factorization (up to order and associates) into irreducible elements, or atoms. Now UFDs can also be characterized by the property that every nonzero nonunit is a product of principal primes or equivalently that every nonzero nonunit has the form $p_1^{\alpha_1} \cdot \ldots \cdot p_n^{\alpha_n}$, where $p_1, \ldots, p_n$ are non-associate principal primes and each $\alpha_i \geq 1$. Each of the $p_i^{\alpha_i}$, in addition to being a power of a prime, has other properties, each of which is subject to generalization. For example, each $p_i^{\alpha_i}$ is primary and the $p_i^{\alpha_i}$ are pairwise comaximal. There exist various generalizations of a (unique) factorization into prime powers in integral domains [Brewer, Heinzer, 2002].

We consider the comaximal factorization introduced by McAdam and Swan [McAdam, Swan, 2004].

They defined a nonzero nonunit element $d$ of an integral domain $D$ to be **pseudo-irreducible** (**pseudo-prime**) if $d = ab$ ($abR \subset dR$) for comaximal $a$ and $b$ implies that $a$ or $b$ is a unit ($aR \subset dR$ or $bR \subset dR$).

A factorization $d = d_1 \ldots d_n$ is a complete comaximal factorization if each $d_i$ is a nonzero nonunit pseudo-irreducible and the $d_i$’s are pairwise comaximal. The integral domain $D$ is a comaximal factorization domain (CFD) if each nonzero nonunit has a complete comaximal factorization.

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Let us start with the following Henriksen example [Henriksen, 1955]

$$R = \{ z_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots \mid z_0 \in \mathbb{Z}, \ a_i \in \mathbb{Q}, \ i = 1, 2, \ldots \}. $$

This domain is a two-dimensional Bezout domain having a unique prime ideal $J(R)$ (Jacobson radical) of height one and having infinitely many maximal ideals corresponding to the maximal ideals of $\mathbb{Z}$. The elements of $J(R)$ are contained in infinitely many maximal ideals of $R$ while the elements of $R \setminus J(R)$ are contained in only finitely many prime ideals.

Henriksen’s Example is a ring of stable range 2 and $R$ is not a ring of stable range 1.

**Definition**

A ring $R$ is said to have a stable range 2 if for every elements $a, b, c \in R$ such that $aR + bR + cR = R$ we have $(a + cx)R + (b + cy)R = R$ for some elements $x, y \in R$. 
Henriksen’s Example is an example of a ring in which every nonzero nonunit element has only finitely many prime ideals minimal over it. By [McAdam, Swan, 2004], Lemma 1.1.(i), in Henriksen’s example every nonzero nonunit element has a complete comaximal factorization. The element of $J(R)$ has a complete comaximal factorization $cx^n$, where $c \in \mathbb{Q}$ ($cx^n$ is a pseudo-irreducible element). By [McAdam, Swan, 2004], Lemma 3.2, any nonzero element of $J(R)$ is pseudo-irreducible. The elements of $R \setminus J(R)$ are contained in only finitely many prime ideals and have a complete comaximal factorization corresponding their factorization in $\mathbb{Z}$.

We will notice $J(R)$ is not generated by $x$, since $\frac{1}{2}x \in J(R)$ and $\frac{1}{2}x \notin xR$.

It is shown in [McAdam, Swan, 2004] that an integral domain $R$ is a complete comaximal factorization if either 1) each nonzero nonunit of $R$ has only finitely many minimal primes or 2) each nonzero nonunit of $R$ is contained in only finitely many maximal ideals.

For a commutative ring $R$, let $\text{spec } R$ and $\text{mspec } R$ denote the collection of all prime ideals and all maximal ideals of $R$, respectively.

The Zariski topology on $\text{spec } R$ is the topology obtained by taking the collection of sets of the form $D(I) = \{ P \in \text{spec } R \mid J \nsubseteq P \}$ (respectively, $V(I) = \{ P \in \text{spec } R \mid J \subseteq P \}$), for every ideal $I$ of $R$ as the open (respectively, closed) sets.
A topological space $X$ is called Noetherian if every nonempty set of closed subsets of $X$ ordered by inclusion has a minimal element. An ideal $J$ of $R$ is called a $J$-radical ideal if it is the intersection of all maximal ideals containing it. Clearly, $J$-radical ideals of $R$ correspond to closed subsets of $mspec R$. 
When considered as a subspace of $space R$, the space $mspace R$ is called a max-spectrum of $R$. So its open and closed subsets are

$$D(I) = D(I) \cap mspec R = \{ M \in mspec R \mid J \not\subseteq M \}$$

and

$$V(I) = V(I) \cap mspec R = \{ M \in mspec R \mid J \subseteq M \},$$

respectively. Clearly max-spectrum of $R$ is $J$-Noetherian if and only if $R$ satisfies the ascending chain condition for $J$-radical ideals, i.e. $R$ is a $J$-Noetherian ring [Estes, Ohm, 1967].

For a commutative $J$-Noetherian Bezout domain $R$ this condition is equivalent to a condition that every nonzero nonunit element of $R$ has only finitely many prime ideals minimal over it [Estes, Ohm, 1967].

Let $R$ be a domain and $a \in R$. Denote by $\text{minspec } a$ the set of prime ideals minimal over $a$.

**Lemma 2.1**

Let $R$ be a Bezout domain and $a$ be a nonzero nonunit element of $R$. Then $a$ is pseudo-irreducible if and only if $\overline{R} = R/aR$ is connected (i.e. its only idempotents are zero and 1).

We will notice that in a local ring, every nonzero nonunit element is pseudo-irreducible.

If $a$ is an element of domain $R$ where $\mid \text{minspec } a \mid = 1$, then $a$ is also pseudo-irreducible.

In Henriksen’s example

$$R = \{ z_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots \mid z_0 \in \mathbb{Z}, \ a_i \in \mathbb{Q}, \ i = 1, 2, \ldots \},$$

$x$ is pseudo-irreducible.
Definition

A commutative ring $R$ is called an **elementary divisor ring** [Kaplansky, 1949] if for an arbitrary matrix $A$ of order $n \times m$ over $R$ there exist invertible matrices $P \in GL_n(R)$ and $Q \in GL_m(R)$ such that

1. $PAQ = D$ is diagonal matrix, $D = (d_{ii})$;
2. $d_{i+1,i+1}R \subseteq d_{ii}R$.

By [Zabavsky, 2009], we have the following result.

**Theorem 2.1**

Every $J$-Noetherian Bezout ring is an elementary divisor ring.

By [Zabavsky, Bokhonko, 2017], for a Bezout domain we have the following result.

**Theorem 2.2**

Let $R$ be a Bezout domain. The following two conditions are equivalent:

1. $R$ is an elementary divisor ring;
2. For any elements $x, y, z, t \in R$ such that $xR + yR = R$ and $zR + tR = R$ there exists an element $\lambda \in R$ such that $x + \lambda y = r \cdot s$, where $rR + zR = R$, $sR + tR = R$ and $rR + sR = R$.

Definition
Let $R$ be a Bezout domain. An element $a \in R$ is called a neat element if for every elements $b, c \in R$ such that $bR + cR = R$ there exist $r, s \in R$ such that $a = rs$ where $rR + bR = R$, $sR + cR = R$ and $rR + sR = R$.

Definition
A Bezout domain is said to be of neat range 1 if for any $c, b \in R$ such that $cR + bR = R$ there exists $t \in R$ such that $a + bt$ is a neat element.
According to Theorem 2.2 we will obtain the following result.

**Theorem 2.3**

A commutative Bezout domain $R$ is an elementary divisor domain if and only if $R$ is a ring of neat range 1.
Theorem 2.4

A nonunit divisor of a neat element of a commutative Bezout domain is a neat element.

Theorem 2.5

Let \( R \) be a \( J \)-Noetherian Bezout domain which is not a ring of stable range 1. Then in \( R \) there exists an element \( a \in R \) such that \( R/aR \) is a local ring.
By [Zabavsky, 2014], any adequate element of a commutative Bezout ring is a neat element.

**Definition**

An element $a$ of a domain $R$ is said to be **adequate**, if for every element $b \in R$ there exist elements $r, s \in R$ such that:

1. $a = rs$;
2. $rR + bR = R$;
3. $s'R + bR \neq R$ for any $s' \in R$ such that $sR \subset s'R \neq R$.

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Definition

A domain $R$ is called adequate if every nonzero element of $R$ is adequate [Larsen, Levis, Shores, 1974].

The most trivial examples of adequate elements are units, atoms in a ring, and also square-free elements [Zabavsky, 2012].


Henriksen observed that in an adequate domain every nonzero prime ideal is contained in an unique maximal ideal [Henriksen, 1955].

**Theorem 2.6**

Let $R$ be a commutative Bezout element and $a$ is non-zero nonunit element of $R$. If $R/aR$ is local ring, then $a$ is an adequate element.

Theorem 2.7
Let $R$ be a $J$-Noetherian Bezout domain which is not a ring of stable range 1. Then in $R$ there exists a nonunit adequate element.

Theorem 2.8
Let $R$ be a Bezout domain in which every nonzero nonunit element has only finitely many prime ideals minimal over it. Then the factor ring $R/aR$ is the finite direct sum of valuation rings.
A minor modification of the proof of Theorem 2.8 gives us the following result.

**Theorem 2.9**

Let $R$ be a commutative Bezout domain in which any nonzero prime ideal is contained in a finite set of maximal ideals. Then for any nonzero element $a \in R$ such that the set $\text{minspec}(aR)$ is finite, the factor ring $\overline{R} = R/aR$ is a finite direct sum of semilocal rings.
A ring $R$ is said to be **everywhere adequate** if any element of $R$ is adequate.

Note that, as shown above, in the case of a commutative ring, which is a finite direct sum of valuation rings, any element of the ring (in particular zero) is adequate, i.e. this ring is everywhere adequate.
A nonzero element $a$ of a ring $R$ is called an **element of almost stable range 1** if the quotient-ring $R/aR$ is a ring of stable range 1.

Any ring of stable range 1 is a ring of almost stable 1 [McGovern, 2007]. But not every element of stable range 1 is an element of almost stable range 1.

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Example

Let $e$ be a nonzero idempotent of a commutative ring $R$ and $eR + aR = R$. Then $ex + ay = 1$ for some elements $x, y \in R$ and $(1 - e)ex + (1 - e)ay = 1 - e$, so $e + a(1 - e)y = 1$. And we have that $e$ is an element of stable range 1 for any commutative ring.

However if you consider the ring $R = \mathbb{Z} \times \mathbb{Z}$ and the element $e = (1, 0) \in R$ then, as shown above, $e$ is an element of stable range 1, by $R/eR \cong \mathbb{Z}$, and $e$ is not an element of almost stable range 1.

Moreover, if $R$ is a commutative principal ideal domain (e.g. ring of integers), which is not of stable range 1, then every nonzero element of $R$ is an element of almost stable range 1.
A commutative ring in which every nonzero element is an element of almost stable range 1 is called a ring of almost stable range 1.

The first example of a ring of almost stable range 1 is a ring of stable range 1. Also, every commutative principal ideal ring which is not a ring of stable range 1 (for example, the ring of integers) is a ring of almost stable range 1 which is not a ring of stable range 1.

We note that a semilocal ring is an example of a ring of stable range 1. Moreover, a direct sum of rings of stable range 1 is a ring of stable range 1.
We obtain the result from the previous theorems.

**Theorem 2.10**

Let $R$ be a Bezout domain in which every nonzero nonunit element has only finitely many prime ideals minimal over it. Then the factor ring $R/aR$ is everywhere adequate if and only if $R$ is a finite direct sum of valuation rings.

**Theorem 2.11**

Let $R$ be a Bezout domain in which every nonzero nonunit element has only finitely many prime ideals minimal over it and any nonzero prime ideal $\text{spec}(aR)$ is contained in a finite set of maximal ideals. Then $a$ is an element of almost stable range 1.
Is it true that every commutative Bezout domain in which any non-zero prime ideal is contained in a finite set of maximal ideals is an elementary divisor ring?
A Bezout ring of stable range 2 which has square stable range 1
The notion of a stable range of a ring was introduced by H. Bass, and became especially popular because of its various applications to the problem of cancellation and substitution of modules. Let us say that a module $A$ satisfies the power-cancellation property if for all modules $B$ and $C$, $A \oplus B \cong A \oplus C$ implies that $B^n \cong C^n$ for some positive integer $n$ (here $B^n$ denotes the direct sum of $n$ copies of $B$). Let us say that a right $R$-module $A$ has the power-substitution property if given any right $R$-module decomposition $M = A_1 \oplus B_1 = A_2 \oplus B_2$ which each $A_i \cong A$, there exist a positive integer $n$ and a submodule $C \subseteq M^n$ such that $M^n = C \oplus B_1^n = C \oplus B_2^n$. 
K. Goodearl pointed out that a commutative ring $R$ has the power-substitution property if and only if $R$ is of (right) power stable range 1, i.e. if $aR + bR = R$ then $(a^n + bx)R = R$ for some $x \in R$ and some integer $n \geq 2$ depending on $a, b \in R$ [Goodearl, 1976].

Recall that a ring $R$ is said to have 1 in the stable range provided that whenever $ax + b = 1$ in $R$, there exists $y \in R$ such that $a + by$ is a unit in $R$. The following Warfield’s theorem shows that 1 in the stable range is equivalent to a substitution property.
Theorem 3.1

Let $A$ be a right $R$-module, and set $E = \text{End}_R(A)$. Then $E$ has 1 in the stable range if and only if for any right $R$-module decomposition $M = A_1 \oplus B_1 = A_2 \oplus B_2$ with each $A_i \cong A$, there exists a submodule $C \subseteq M$ such that $M = C \oplus B_1 = C \oplus B_2$. [Goodearl, 1976]

A ring $R$ is said to have 2 in the stable range if for any $a_1, \ldots, a_r \in R$ where $r \geq 3$ such that $a_1 R + \cdots + a_r R = R$, there exist elements $b_1, \ldots, b_{r-1} \in R$ such that

$$(a_1 + a_r b_1)R + (a_2 + a_r b_2)R + \cdots + (a_{r-1} + a_r b_{r-1})R = R.$$
K. Goodearl pointed out to us the following result.

**Proposition 3.1**

Let $R$ be a commutative ring which has 2 in the stable range. If $R$ satisfies right power-substitution, then so does $M_n(R)$, for all $n$. [Goodearl, 1976]

Our goal this paper is to study certain algebraic versions of the notion of stable range 1. In this paper we study a Bezout ring which has 2 in the stable range and which is a ring square stable range 1.
A ring $R$ is said to have (right) **square stable range 1** (written $ssr(R) = 1$) if $aR + bR = R$ for any $a, b \in R$ implies that $a^2 + bx$ is an invertible element of $R$ for some $x \in R$. 
Considering the problem of factorizing the matrix \( \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \) into a product of two Toeplitz matrices. D. Khurana, T.Y. Lam and Zhou Wang were led to ask go units of the form \( a^2 + bx \) given that \( aR + bR = R \).

Obviously, a commutative ring which has 1 in the stable range is a ring which has (right) square stable range 1, but not vice versa in general. Examples of rings which have (right) square stable range 1 are rings of continuous real-valued functions on topological spaces and real holomorphy rings in formally real fields [Khurana, Lam, Wang, 2011].

Proposition 3.2

For any ring $R$ with $ssr(R) = 1$, we have that $R$ is right quasi-duo (i.e. $R$ is a ring in which every maximal right ideal is an ideal). [Khurana, Lam, Wang, 2011]

We say that matrices $A$ and $B$ over a ring $R$ are equivalent if there exist invertible matrices $P$ and $Q$ of appropriate sizes such that $B = PAQ$. If for a matrix $A$ there exists a diagonal matrix $D = \text{diag}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r, 0, \ldots, 0)$ such that $A$ and $D$ are equivalent and $R\varepsilon_{i+1}R \subseteq \varepsilon_i R \cap R\varepsilon_i$ for every $i$ then we say that the matrix $A$ has a canonical diagonal reduction. A ring $R$ is called an elementary divisor ring if every matrix over $R$ has a canonical diagonal reduction.
If every $(1 \times 2)$-matrix($(2 \times 1)$-matrix) over a ring $R$ has a canonical diagonal reduction then $R$ is called a **right (left) Hermitian ring**. A ring which is both right and left Hermitian is called an **Hermitian ring**. Obviously, a commutative right (left) Hermitian ring is an Hermitian ring. We note that a right Hermitian ring is a ring in which every finitely generated right ideal is principal.
Theorem 3.2
Let $R$ be a right quasi-duo elementary divisor ring. Then for any $a \in R$ there exists an element $b \in R$ such that $RaR = bR = Rb$. If in addition all zero-divisors of $R$ lie in the Jacobson radical, then $R$ is a duo ring. [Zabavsky, Komarnytskii, 1990]

Recall that a right (left) duo ring is a ring in which every right (left) ideal is two-sided. A duo ring is a ring which is both left and right duo ring.
We have proved the next result.

**Theorem 3.3**

Let $R$ be an elementary divisor ring which has (right) square stable range 1 and which all zero-divisors of $R$ lie in Jacobson radical of $R$, then $R$ is a duo ring.
Proposition 3.3

Let $R$ be a Hermitian duo ring. For every $a, b, c \in R$ such that $aR + bR + cR = R$ the following conditions are equivalent:

1) there exist elements $p, q \in R$ such that $paR + (pb + qc)R = R$;

2) there exist elements $\lambda, u, v \in R$ such that $b + \lambda c = vu$, where $uR + aR = R$ and $vR + cR = R$. 
Remark 3.1

In Proposition 3.3 we can choose the elements $u$ and $v$ such that $uR + vR = R$. 
Proposition 3.4

Let $R$ be an Hermitian duo ring. Then the following conditions are equivalent:

1) $R$ is an elementary divisor duo ring;

2) for every $x, y, z, t \in R$ such that $xR + yR = R$ and $zR + tR = R$ there exists an element $\lambda \in R$ such that $x + \lambda y = vu$, where $vR + zR = R$ and $uR + tR = R$. 
Definition

Let $R$ be a duo ring. We say that an element $a \in R \setminus \{0\}$ is **neat** if for any elements $b, c \in R$ such that $bR + cR = R$ there exist elements $r, s \in R$ such that $a = rs$, where $rR + bR = R$, $sR + cR = R$, $rR + sR = R$. 
Definition

We say that a duo ring $R$ has **neat range 1** if for every $a, b \in R$ such that $aR + bR = R$ there exists an element $t \in R$ such that $a + bt$ is a neat element.
According to Propositions 3.3, 3.4 and Remark 3.1 we have the following result.

**Theorem 3.4**

A Hermitian duo ring $R$ is an elementary divisor ring if and only if $R$ has neat range 1.
The term “neat range 1” substantiates the following theorem.

**Theorem 3.5**

Let $R$ be a Hermitian duo ring. If $c$ is a neat element of $R$ then $R/cR$ is a clean ring.
Taking into account the Theorem 3.3 and Theorem 3.4 we have the following result.

**Theorem 3.5**

A Hermitian ring $R$ which has (right) square stable range 1 is an elementary divisor ring if and only if $R$ is a duo ring of neat range 1.
Let $R$ be a commutative Bezout ring. The matrix $A$ of order 2 over $R$ is said to be a **Toeplitz matrix** if it is of the form

\[
\begin{pmatrix}
a & b \\
c & a
\end{pmatrix}
\]

where $a, b, c \in R$. 

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Notice that if $A$ is an invertible Toeplitz matrix, then $A^{-1}$ is also an invertible Toeplitz matrix.

**Definition**

A commutative Hermitian ring $R$ is called a **Toeplitz ring** if for any $a, b \in R$ there exist an invertible Toeplitz matrix $T$ such that $(a, b) T = (d, 0)$ for some element $d \in R$. 
Theorem 3.7

A commutative Hermitian ring $R$ is a Toeplitz ring if and only if $R$ is a ring of (right) square range 1.
Theorem 3.8

Let \( R \) be a commutative ring of square stable range 1. Then for any row \((a, b)\), where \(aR + bR = R\), there exists an invertible Toeplitz matrix

\[
T = \begin{pmatrix} a & b \\ x & a \end{pmatrix},
\]

where \( x \in R \).
Recall that $GE_n(R)$ denotes a group of $n \times n$ elementary matrices over ring $R$. The following theorem demonstrated that it is sufficient to consider only the case of matrices of order 2 in Theorem 3.7.

**Theorem 3.9**

Let $R$ be a commutative elementary divisor ring. Then for any $n \times m$ matrix $A$ ($n > 2$, $m > 2$) one can find matrices $P \in GE_n(R)$ and $Q \in GE_m(R)$ such that

$$PAQ = \begin{pmatrix} e_1 & 0 & \ldots & 0 & 0 \\ 0 & e_2 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & e_s & 0 \\ 0 & 0 & \ldots & 0 & A_0 \end{pmatrix}$$

where $e_i$ is a divisor of $e_{i+1}$, $1 \leq i \leq s - 1$, and $A_0$ is a $2 \times k$ or $k \times 2$ matrix for some $k \in \mathbb{N}$. [Zabavsky, 2012]

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Theorem 3.10

Let $R$ be a commutative elementary divisor ring of (right) square stable range 1. Then for any $2 \times 2$ matrix $A$ one can find invertible Toeplitz matrices $P$ and $Q$ such that

$$PAQ = \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix},$$

where $e_i$ is a divisor of $e_2$. 
Is it true that every commutative Bezout domain of stable range 2 which has (right) square stable range 1 is an elementary divisor ring?
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