J-Noetherian Bezout domain which are not of stable range 1. A Bezout ring of stable range 2 which has square stable range 1

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May 2, 2019

J-Noetherian Bezout domain which are not of stable range 1

All rings considered will be commutative and have identity.

The notion of a stable range was useful in modern research on theory of diagonalization of matrices.

Recall that a ring R is a **ring of stable range 1** if for any $a, b \in R$ such that $aR + bR = R$ we have $(a + bt)R = R$ for some $t \in R$.

Studying an elementary divisor ring W. McGovern has introduced the concept of ring of almost stable range 1 as a ring whose proper homomorphic images all have stable range 1 [McGovern,2007].

By [McGovern,2007] a ring of stable range 1 is a ring of almost stable range 1. At the same time not every element of stable range 1 is an element of almost stable range 1.

W. McGovern, Bezout rings with almost stable range 1 are elementary divisor rings, *J. Pure Appl. Algebra* **212** (2007) 340–348.

Recall that element a is an **element of stable range 1** if for any $b \in R$ such that $aR + bR = R$ we have $a + bt$ is an invertible element.

An **element** $a \in R$ is of **almost stable range 1** if R/aR is a ring of stable range 1.

If $R = \mathbb{Z} \times \mathbb{Z}$, then $e = (1, 0)$ is element of stable range 1, but $R/eR \cong \mathbb{Z}$ is not a ring of stable range 1.

The problem of finding the element of almost stable range 1 in rings which are not rings of stable range 1 is required, in accordance with the above and considerations in [Zabavsky, 2017]. In this paper on the basis of theory comaximal factorization we prove that in any J-Noetherian Bezout ring which are not of stable range 1 there exist a nonunit element of almost stable range 1.

B. V. Zabavsky, Conditions for stable range of an elementary divisor rings, *Comm, Algebra* **45**(9) (2017) 4062–4066.

Unique factorization domains are, of course, integral domains in which every nonzero nonunit element has a unique factorization (up to order and associates) into irreducible elements, or atoms. Now UFDs can also be characterized by the property that every nonzero nonunit is a product of principal primes or equivalently that every nonzero nonunit has the form $p_1^{\alpha_1} \cdot \ldots \cdot p_n^{\alpha_n}$, where $p_1, \ldots, p_n$ are non-associate principal primes and each $\alpha_i \geq 1$. Each of the $p_i^{\alpha_i}$, in addition to being a power of a prime, has other properties, each of which is subject to generalization. For example, each $p_i^{\alpha_i}$ is primary and the $p_i^{\alpha_i}$ are pairwise comaximal. There exist various generalizations of a (unique) factorization into prime powers in integral domains [Brewer, Heinzer, 2002].

J. W. Brewer and W. J. Heinzer, On decomposing ideals into products of comaximal ideals, *Comm. Algebra* **30**(12) (2002) 5999–6010.

We consider the comaximal factorization introduced by McAdam and Swan [McAdam, Swan, 2004].

They defined a nonzero nonunit element d of an integral domain D to be **pseudo-irreducible** (**pseudo-prime**) if $d = ab$ ($abR \subset dR$) for comaximal a and b implies that a or b is a unit ($aR \subset dR$ or $bR \subset dR$).

A factorization $d = d_i \ldots d_n$ is a complete comaximal factorization if each d_i is a nonzero nonunit pseudo-irreducible and the d_i's are pairwise comaximal. The integral domain D is a comaximal factorization domain (CFD) if each nonzero nonunit has a complete comaximal factorization.

S. McAdam and R. Swan, Unique comaximal factorization, *J. Algebra* **276** (2004) 180–192.

Let us start with the following Henriksen example [Henriksen, 1955]

$$R = \{z_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots \mid z_0 \in \mathbb{Z},\ a_i \in \mathbb{Q},\ i = 1, 2, \ldots\}.$$

This domain is a two-dimensional Bezout domain having a unique prime ideal $J(R)$ (Jacobson radical) of height one and having infinitely many maximal ideals corresponding to the maximal ideals of $\mathbb{Z}$. The elements of $J(R)$ are contained in infinitely many maximal ideals of R while the elements of $R \setminus J(R)$ are contained in only finitely many prime ideals.

M. Henriksen, Some remarks about elementary divisor rings, *Michigan Math. J.* **3** (1955/56) 159–163.

Henriksen's Example is a ring of stable range 2 and R is not a ring of stable range 1.

Definition

A ring R is said to have a stable range 2 if for every elements $a, b, c \in R$ such that $aR + bR + cR = R$ we have $(a + cx)R + (b + cy)R = R$ for some elements $x, y \in R$.

Henriksen's Example is an example of a ring in which every nonzero nonunit element has only finitely many prime ideals minimal over it. By [McAdam, Swan, 2004], Lemma 1.1.(i), in Henriksen's example every nonzero nonunit element has a complete comaximal factorization. The element of $J(R)$ has a complete comaximal factorization cx^n, where $c \in \mathbb{Q}$ (cx^n is a pseudo-irreducible element). By [McAdam, Swan, 2004], Lemma 3.2, any nonzero element of $J(R)$ is pseudo-irreducible. The elements of $R \setminus J(R)$ are contained in only finitely many prime ideals and have a complete comaximal factorization corresponding their factorization in $\mathbb{Z}$. We will notice $J(R)$ is not generated by x, since $\frac{1}{2}x \in J(R)$ and $\frac{1}{2}x \notin xR$.

S. McAdam and R. Swan, Unique comaximal factorization, *J. Algebra* **276** (2004) 180–192.

It is shown in [McAdam, Swan, 2004] that an integral domain R is a complete comaximal factorization if either 1) each nonzero nonunit of R has only finitely many minimal primes or 2) each nonzero nonunit of R is contained in only finitely many maximal ideals.

S. McAdam and R. Swan, Unique comaximal factorization, *J. Algebra* **276** (2004) 180–192.

For a commutative ring R, let *spec* R and *mspec* R denote the collection of all prime ideals and all maximal ideals of R, respectively.

The Zariski topology on *spec* R is the topology obtained by taking the collection of sets of the form $D(I) = \{P \in spec\, R \mid J \nsubseteq P\}$ (respectively, $V(I) = \{P \in spec\, R \mid J \subseteq P\}$), for every ideal I of R as the open (respectively, closed) sets.

A topological *space* X is called Noetherian if every nonempty set of closed subsets of X ordered by inclusion has a minimal element. An ideal J of R is called a J-radical ideal if it is the intersection of all maximal ideals containing it. Clearly, J-radical ideals of R correspond to closed subsets of *mspec* R.

When considered as a subspace of *space* R, the space *mspace* R is called a max-spectrum of R. So its open and closed subsets are

$$D(I) = D(I) \cap mspec\, R = \{M \in mspec\, R \mid J \not\subseteq M\}$$

and

$$V(I) = V(I) \cap mspec\, R = \{M \in mspec\, R \mid J \subseteq M\},$$

respectively. Clearly max-spectrum of R is J-Noetherian if and only if R satisfies the ascending chain condition for J-radical ideals, i.e. R is a J-Noetherian ring [Estes, Ohm, 1967].

D. Estes and J. Ohm, Stable range in commutative rings, *J. Algebra* **7** (1967) 343–362.

For a commutative J-Noetherian Bezout domain R this condition is equivalent to a condition that every nonzero nonunit element of R has only finitely many prime ideals minimal over it [Estes, Ohm, 1967].

Let R be a domain and $a \in R$. Denote by *minspec* a the set of prime ideals minimal over a.

Lemma 2.1

Let R be a Bezout domain and a be a nonzero nonunit element of R. Then a is pseudo-irreducible if and only if $\overline{R} = R/aR$ is connected (i.e. its only idempotents are zero and 1).

D. Estes and J. Ohm, Stable range in commutative rings, *J. Algebra* **7** (1967) 343–362.

We will notice that in a local ring, every nonzero nonunit element is pseudo-irreducible.

If a is a element of domain R where $|\, minspec\, a \,| = 1$, then a is also pseudo-irreducible.

In Henriksen's example

$$R = \{z_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots \mid z_0 \in \mathbb{Z},\ a_i \in \mathbb{Q},\ i = 1, 2, \ldots\},$$

x is pseudo-irreducible.

Definition

A commutative ring R is called an **elementary divisor ring** [Kaplansky,1949] if for an arbitrary matrix A of order $n \times m$ over R there exist invertible matrices $P \in GL_n(R)$ and $Q \in GL_m(R)$ such that

(1) $PAQ = D$ is diagonal matrix, $D = (d_{ii})$;

(2) $d_{i+1,i+1}R \subset d_{ii}R$.

I. Kaplansky, Elementary divisors and modules, *Trans. Amer. Math. Soc.* **66** (1949) 464–491.

By [Zabavsky,2009], we have the following result.

Theorem 2.1

Every J-Noetherian Bezout ring is an elementary divisor ring.

B. V. Zabavsky, Fractionally regular Bezout rings, *Mat. Stud.* **32**(1) (2009) 76–80.

By [Zabavsky, Bokhonko, 2017], for a Bezout domain we have the following result.

Theorem 2.2

Let R be a Bezout domain. The following two condition are equivalent:

(1) R is an elementary divisor ring;

(2) for any elements $x, y, z, t \in R$ such $xR + yR = R$ and $zR + tR = R$ there exists an element $\lambda \in R$ such that $x + \lambda y = r \cdot s$, where $rR + zR = R$, $sR + tR = R$ and $rR + sR = R$.

B. V. Zabavsky and V. V. Bokhonko, A criterion of elementary divisor domain for distributive domains, *Algebra and Discrete Math.* **23**(1) (2017) 1–6.

Definition

Let R be a Bezout domain. An element $a \in R$ is called a **neat element** if for every elements $b, c \in R$ such that $bR + cR = R$ there exist $r, s \in R$ such that $a = rs$ where $rR + bR = R$, $sR + cR = R$ and $rR + sR = R$.

Definition

A Bezout domain is said to be of **neat range 1** if for any $c, b \in R$ such that $cR + bR = R$ there exists $t \in R$ such that $a + bt$ is a neat element.

According to Theorem 2.2 we will obtain the following result.

Theorem 2.3

A commutative Bezout domain R is an elementary divisor domain if and only if R is a ring of neat range 1.

Theorem 2.4

A nonunit divisor of a neat element of a commutative Bezout domain is a neat element.

Theorem 2.5

Let R be a J-Noetherian Bezout domain which is not a ring of stable range 1. Then in R there exists an element $a \in R$ such that R/aR is a local ring.

By [Zabavsky, 2014], any adequate element of a commutative Bezout ring is a neat element.

Definition

An element a of a domain R is said to be **adequate** , if for every element $b \in R$ there exist elements $r, s \in R$ such that:

(1) $a = rs$;

(2) $rR + bR = R$;

(3) $s'R + bR \neq R$ for any $s' \in R$ such that $sR \subset s'R \neq R$.

B. V. Zabavsky, Diagonal reduction of matrices over finite stable range rings, *Mat. Stud.* **41** (2014) 101–108.

Definition

A domain R is called **adequate** if every nonzero element of R is adequate [Larsen, Levis, Shores, 1974].

The most trivial examples of adequate elements are units, atoms in a ring, and also square-free elements [Zabavsky, 2012].

M. Larsen, W. Levis and T. Shores, Elementary divisor rings and finitely presented modules, *Trans. Amer. Math. Soc.* **187** (1974) 231–248.

B. V. Zabavsky, *Diagonal reduction of matrices over rings* (Mathematical Studies, Monograph Series, v. XVI, VNTL Publishers, Lviv, 2012).

Henriksen observed that in an adequate domain every nonzero prime ideal is contained in an unique maximal ideal [Henriksen, 1955].

Theorem 2.6

Let R be a commutative Bezout element and a is non-zero nonunit element of R. If R/aR is local ring, then a is an adequate element.

M. Henriksen, Some remarks about elementary divisor rings, *Michigan Math. J.* **3** (1955/56) 159–163.

Theorem 2.7

Let R be a J-Noetherian Bezout domain which is not a ring of stable range 1. Then in R there exists a nonunit adequate element.

Theorem 2.8

Let R be a Bezout domain in which every nonzero nonunit element has only finitely many prime ideals minimal over it. Then the factor ring R/aR is the finite direct sum of valuation rings.

A minor modification of the proof of Theorem 2.8 gives us the following result.

Theorem 2.9

Let R be a commutative Bezout domain in which any nonzero prime ideal is contained in a finite set of maximal ideals. Then for any nonzero element $a \in R$ such that the set $minspec(aR)$ is finite, the factor ring $\overline{R} = R/aR$ is a finite direct sum of semilocal rings.

Definition

A ring R is said to be **everywhere adequate** if any element of R is adequate.

Note that, as shown above, in the case of a commutative ring, which is a finite direct sum of valuation rings, any element of the ring (in particular zero) is adequate, i.e. this ring is everywhere adequate.

Definition

A nonzero element a of a ring R is called an **element of almost stable range 1** if the quotient-ring R/aR is a ring of stable range 1.

Any ring of stable range 1 is a ring of almost stable 1 [McGovern,2007]. But not every element of stable range 1 is an element of almost stable range 1.

W. McGovern, Bezout rings with almost stable range 1 are elementary divisor rings, *J. Pure Appl. Algebra* **212** (2007) 340–348.

Example

Let e be a nonzero idempotent of a commutative ring R and $eR + aR = R$. Then $ex + ay = 1$ for some elements $x, y \in R$ and $(1-e)ex + (1-e)ay = 1-e$, so $e + a(1-e)y = 1$. And we have that e is an element of stable range 1 for any commutative ring.
However if you consider the ring $R = \mathbb{Z} \times \mathbb{Z}$ and the element $e = (1,0) \in R$ then, as shown above, e is an element of stable range 1, by $R/eR \cong \mathbb{Z}$, and e is not an element of almost stable range 1.
Moreover, if R is a commutative principal ideal domain (e.g. ring of integers), which is not of stable range 1, then every nonzero element of R is an element of almost stable range 1.

Definition

A commutative ring in which every nonzero element is an **element of almost stable range 1** is called a ring of almost stable range 1.

The first example of a ring of almost stable range 1 is a ring of stable range 1. Also, every commutative principal ideal ring which is not a ring of stable range 1 (for example, the ring of integers) is a ring of almost stable range 1 which is not a ring of stable range 1.

We note that a semilocal ring is an example of a ring of stable range 1. Moreover, a direct sum of rings of stable range 1 is a ring of stable range 1.

We obtain the result from the previous theorems.

Theorem 2.10

Let R be a Bezout domain in which every nonzero nonunit element has only finitely many prime ideals minimal over it. Then the factor ring R/aR is everywhere adequate if and only if R is a finite direct sum of valuation rings.

Theorem 2.11

Let R be a Bezout domain in which every nonzero nonunit element has only finitely many prime ideals minimal over it and any nonzero prime ideal $spec(aR)$ is contained in a finite set of maximal ideals. Then a is an element of almost stable range 1.

Open Question

Is it true that every commutative Bezout domain in which any non-zero prime ideal is contained in a finite set of maximal ideals is an elementary divisor ring?

A Bezout ring of stable range 2 which has square stable range 1

The notion of a stable range of a ring was introduced by H. Bass, and became especially popular because of its various applications to the problem of cancellation and substitution of modules. Let us say that a module A satisfies the power-cancellation property if for all modules B and C, $A \oplus B \cong A \oplus C$ implies that $B^n \cong C^n$ for some positive integer n (here B^n denotes the direct sum of n copies of B). Let us say that a right R-module A has the power-substitution property if given any right R-module decomposition $M = A_1 \oplus B_1 = A_2 \oplus B_2$ which each $A_i \cong A$, there exist a positive integer n and a submodule $C \subseteq M^n$ such that $M^n = C \oplus B_1^n = C \oplus B_2^n$.

K. Goodearl pointed out that a commutative rind R has the power-substitution property if and only if R is of (right) power stable range 1, i.e. if $aR + bR = R$ than $(a^n + bx)R = R$ for some $x \in R$ and some integer $n \geq 2$ depending on $a, b \in R$ [Goodearl, 1976].

K. R. Goodearl, Power-cancellation of groups and modules, *Pacific J. Math.* **64** (1976) 487–411.

Recall that a ring R is said to have 1 in the stable range provided that whenever $ax + b = 1$ in R, there exists $y \in R$ such that $a + by$ is a unit in R. The following Warfield's theorem shows that 1 in the stable range is equivalent to a substitution property.

Theorem 3.1

Let A be a right R-module, and set $E = \mathrm{End}_R(A)$. Then E has 1 in the stable range if and only if for any right R-module decomposition $M = A_1 \oplus B_1 = A_2 \oplus B_2$ with each $A_i \cong A$, there exists a submodule $C \subseteq M$ such that $M = C \oplus B_1 = C \oplus B_2$. [Goodearl, 1976]

K. R. Goodearl, Power-cancellation of groups and modules, *Pacific J. Math*. **64** (1976) 487–411.

A ring R is said to have 2 in the stable range if for any $a_1, \dots, a_r \in R$ where $r \geq 3$ such that $a_1R + \dots + a_rR = R$, there exist elements $b_1, \dots, b_{r-1} \in R$ such that

$$(a_1 + a_rb_1)R + (a_2 + a_rb_2)R + \dots + (a_{r-1} + a_rb_{r-1})R = R.$$

K. Goodearl pointed out to us the following result.

Proposition 3.1

Let R be a commutative ring which has 2 in the stable range. If R satisfies right power-substitution, then so does $M_n(R)$, for all n. [Goodearl, 1976]

K. R. Goodearl, Power-cancellation of groups and modules, *Pacific J. Math.* **64** (1976) 487–411.

Our goal this paper is to study certain algebraic versions of the notion of stable range 1. In this paper we study a Bezout ring which has 2 in the stable range and which is a ring square stable range 1.

A ring R is said to have (right) **square stable range 1** (written $ssr(R) = 1$) if $aR + bR = R$ for any $a, b \in R$ implies that $a^2 + bx$ is an invertible element of R for some $x \in R$.

Considering the problem of factorizing the matrix $\left(\begin{smallmatrix} a & 0 \\ b & 0 \end{smallmatrix}\right)$ into a product of two Toeplitz matrices. D. Khurana, T.Y. Lam and Zhou Wang were led to ask go units of the form $a^2 + bx$ given that $aR + bR = R$.

Obviously, a commutative ring which has 1 in the stable range is a ring which has (right) square stable range 1, but not vice versa in general. Examples of rings which have (right) square stable range 1 are rings of continuous real-valued functions on topological spaces and real holomorphy rings in formally real fields [Khurana, Lam, Wang, 2011].

D. Khurana, T. Y. Lam and Zh. Wang, Rings of square stable range one, *J. Algebra* **338** (2011) 122–143.

Proposition 3.2

For any ring R with $ssr(R) = 1$, we have that R is right quasi-duo (i.e. R is a ring in which every maximal right ideal is an ideal). [Khurana, Lam, Wang, 2011]

D. Khurana, T. Y. Lam and Zh. Wang, Rings of square stable range one, *J. Algebra* **338** (2011) 122–143.

We say that matrices A and B over a ring R are **equivalent** if there exist invertible matrices P and Q of appropriate sizes such that $B = PAQ$. If for a matrix A there exists a diagonal matrix $D = diag(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r, 0, \ldots, 0)$ such that A and D are equivalent and $R\varepsilon_{i+1}R \subseteq \varepsilon_i R \cap R\varepsilon_i$ for every i then we say that the matrix A has a **canonical diagonal reduction**. A ring R is called an **elementary divisor ring** if every matrix over R has a canonical diagonal reduction.

If every (1×2)-matrix ((2×1)-matrix) over a ring R has a canonical diagonal reduction then R is called a **right (left) Hermitian ring**. A ring which is both right and left Hermitian is called an **Hermitian ring**. Obviously, a commutative right (left) Hermitian ring is an Hermitian ring. We note that a right Hermitian ring is a ring in which every finitely generated right ideal is principal.

Theorem 3.2

Let R be a right quasi-duo elementary divisor ring. Then for any $a \in R$ there exists an element $b \in R$ such that $RaR = bR = Rb$. If in addition all zero-divisors of R lie in the Jacobson radical, then R is a duo ring. [Zabavsky, Komarnytskii, 1990]

B. V. Zabavsky and M. Ya. Komarnytskii, Distributive elementary divisor domains, *Ukr. Math. J.* **42** (1990) 890–892.

Recall that a right (left) duo ring is a ring in which every right (left) ideal is two-sided. A duo ring is a ring which is both left and right duo ring.

We have proved the next result.

Theorem 3.3

Let R be an elementary divisor ring which has (right) square stable range 1 and which all zero-divisors of R lie in Jacobson radical of R, then R is a duo ring.

Proposition 3.3

Let R be a Hermitian duo ring. For every $a, b, c \in R$ such that $aR + bR + cR = R$ the following conditions are equivalent:

1) there exist elements $p, q \in R$ such that $paR + (pb + qc)R = R$;
2) there exist elements $\lambda, u, v \in R$ such that $b + \lambda c = vu$, where $uR + aR = R$ and $vR + cR = R$.

Remark 3.1

In Proposition 3.3 we can choose the elements u and v such that $uR + vR = R$.

Proposition 3.4

Let R be an Hermitian duo ring. Then the following conditions are equivalent:

1) R is an elementary divisor duo ring;
2) for every $x, y, z, t \in R$ such that $xR + yR = R$ and $zR + tR = R$ there exists an element $\lambda \in R$ such that $x + \lambda y = vu$, where $vR + zR = R$ and $uR + tR = R$.

Definition

Let R be a duo ring. We say that an element $a \in R \backslash \{0\}$ is **neat** if for any elements $b, c \in R$ such that $bR + cR = R$ there exist elements $r, s \in R$ such that $a = rs$, where $rR + bR = R$, $sR + cR = R$, $rR + sR = R$.

Definition

We say that a duo ring R has **neat range 1** if for every $a, b \in R$ such that $aR + bR = R$ there exists an element $t \in R$ such that $a + bt$ is a neat element.

According to Propositions 3.3, 3.4 and Remark 3.1 we have the following result.

Theorem 3.4

A Hermitian duo ring R is an elementary divisor ring if and only if R has neat range 1.

The term "neat range 1" substantiates the following theorem.

Theorem 3.5

Let R be a Hermitian duo ring. If c is a neat element of R then R/cR is a clean ring.

Taking into account the Theorem 3.3 and Theorem 3.4 we have the following result.

Theorem 3.5

A Hermitian ring R which has (right) square stable range 1 is an elementary divisor ring if and only if R is a duo ring of neat range 1.

Let R be a commutative Bezout ring. The matrix A of order 2 over R is said to be a **Toeplitz matrix** if it is of the form

$$\begin{pmatrix} a & b \\ c & a \end{pmatrix}$$

where $a, b, c \in R$.

Notice that if A is an invertible Toeplitz matrix, then A^{-1} is also an invertible Toeplitz matrix.

Definition

A commutative Hermitian ring R is called a **Toeplitz ring** if for any $a, b \in R$ there exist an invertible Toeplitz matrix T such that $(a, b)T = (d, 0)$ for some element $d \in R$.

Theorem 3.7

A commutative Hermitian ring R is a Toeplitz ring if and only if R is a ring of (right) square range 1.

Theorem 3.8

Let R be a commutative ring of square stable range 1. Then for any row (a, b), where $aR + bR = R$, there exists an invertible Toeplitz matrix

$$T = \begin{pmatrix} a & b \\ x & a \end{pmatrix},$$

where $x \in R$.

Recall that $GE_n(R)$ denotes a group of $n \times n$ elementary matrices over ring R. The following theorem demonstrated that it is sufficient to consider only the case of matrices of order 2 in Theorem 3.7.

Theorem 3.9

Let R be a commutative elementary divisor ring. Then for any $n \times m$ matrix A ($n > 2$, $m > 2$) one can find matrices $P \in GE_n(R)$ and $Q \in GE_m(R)$ such that

$$PAQ = \begin{pmatrix} e_1 & 0 & \dots & 0 & 0 \\ 0 & e_2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & e_s & 0 \\ 0 & 0 & \dots & 0 & A_0 \end{pmatrix}$$

where e_i is a divisor of e_{i+1}, $1 \leq i \leq s-1$, and A_0 is a $2 \times k$ or $k \times 2$ matrix for some $k \in \mathbb{N}$. [Zabavsky, 2012]

B. V. Zabavsky *Diagonal reduction of matrices over rings* (Mathematical Studies, Monograph Series, volume XVI, VNTL Publishers, Lviv, 2012).

Theorem 3.10

Let R be a commutative elementary divisor ring of (right) square stable range 1. Then for any 2×2 matrix A one can find invertible Toeplitz matrices P and Q such that

$$PAQ = \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix},$$

where e_i is a divisor of e_2.

Open Question

Is it true that every commutative Bezout domain of stable range 2 which has (right) square stable range 1 is an elementary divisor ring?

The End