Bass stable range

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Stable range conditions

The notion of stable range for rings was introduced by H. Bass. This notion was used by Bass for the study of the stability properties of linear groups in algebraic K-theory, but later it become an important notion in ring theory in its own right.

Definition

Let R be an associative ring with identity and $1 \neq 0$. If for a row $(a_1, \ldots, a_n) \in R^n$ one can find $(b_1, \ldots, b_n) \in R^n$ such that

$$a_1 b_1 + \ldots + a_n b_n = 1,$$

we say that $(a_1, \ldots, a_n)$ is a *right unimodular row* over R. The notion of *a left unimodular row* over R can be defined similarly. Also, the number n is called the length of the unimodular row.

Clearly one can say that a nonzero row $(a_1, \ldots, a_n)$ is left (right) unimodular if and only if it generates a trivial non zero left (right) ideal.

Definition

We say that a positive integer n is in the *stable range* [a] of a ring R if for every right unimodular row $(a_1, \ldots, a_{n+1}) \in R^{n+1}$ there exists $(b_1, \ldots, b_{n+1}) \in R^{n+1}$ such that

$$a_1 b_1 + \ldots + a_{n+1} b_{n+1} = 1$$

and $(b_1, \ldots, b_n)$ is a left unimodular row. In such case we also say that n is a stable range of the ring R, or R satisfies the n-stable range condition, and write that as $\mathrm{st.r.}(R) = n$. If there is no such n then we say that the stable range of R is infinite and write this fact as $\mathrm{st.r.}(R) = \infty$.

[a] Bass H. *K-theory and stable algebra* // Inst. Hautes Etudes. Sci. Publ. Math. – 1964. – 22. – P. 485-544.

It is natural to say that the stable range of R is the least possible positive integer n which satisfies the stable range condition for R, but keep in mind that, in fact, any bigger m also satisfies this condition, i.e.

$$\mathrm{st.r.}(R) = \{n, n+1, \dots\}.$$

Proposition

For a ring R the following conditions are equivalent:

(i) *for every right unimodular row $(a_1, \dots, a_{n+1}) \in R^{n+1}$ there exists $(b_1, \dots, b_{n+1}) \in R^{n+1}$ such that*

$$a_1 b_1 + \ldots + a_{n+1} b_{n+1} = 1, \; Rb_1 + \ldots + Rb_n = R;$$

(ii) *for every right unimodular row $(a_1, \dots, a_{n+1}) \in R^{n+1}$ there exists $(x_1, \dots, x_n) \in R^n$ such that*

$$(a_1, \dots, a_n) + a_{n+1}(x_1, \dots, x_n) = (a_1 + a_{n+1}x_1, \dots, a_n + a_{n+1}x_n)$$

is right unimodular.

Since the definition of a stable range is given for the case of right unimodular rows then it is natural to use the term “right stable range” emphasising on the type of unimodular rows we use, and define the “left stable range” in a similar manner. However, due to the results of Vasserstein [1], [2] and Warfield [3] this notions coincide and we simply speak about the “stable range” of a ring R.

[1] Vaserstein L. N. *A stable range of rings and dimension of topological spaces* // Funk. An. Pril. – 1971. – 5(2). – P. 17-27

[2] Vaserstein L. N. *Bass's first stable range condition* // J. Pure Appl. Alg.– 1984. – 34(2-3). – P. 319-330

[3] Warfield Jr. R. B. *Stable equivalence of matrices and resolutions* // Comm. Algebra. – 1978. – 6(17). – P. 1811-1828.

Theorem (Stable range left-right symmetry)

Let R be a ring. Then

$$\mathrm{st.r.}(R) = \mathrm{st.r.}(R^{op}),$$

where R^{op} means the opposite ring to R.

Now we have a two results that state that stable range of quotient-ring R/I cannot exceed the stable range of R, and they coincide when I is a radical ideal.

Proposition

If I is a two-sided ideal of R then $\mathrm{st.r.}(R/I) \leq \mathrm{st.r.}(R)$.

Theorem

For any ring R and any ideal $I \subseteq J(R)$:

$$\mathrm{st.r.}(R) = \mathrm{st.r.}(R/I).$$

In particular, $\mathrm{st.r.}(R) = \mathrm{st.r.}(R/J(R))$.

We are going to derive the exact formula that allows us to calculate the stable range of matrix ring over R based on the stable range of R itself. We follow the method which was first used by Vaserstein [[4]].

[4] Vaserstein L. N. *A stable range of rings and dimension of topological spaces* // Funk. An. Pril. – 1971. – 5(2). – P. 17-27

Theorem (Vaserstein's formula)

For any ring R and any $k \geq 1$:

$$\mathrm{st.r.}(M_k(R)) = 1 + \left\lceil \frac{\mathrm{st.r.}(R) - 1}{k} \right\rceil,$$

where $\lceil x \rceil$ denotes the least integer greater than or equal to a real number x.

Definition

Suppose that P is a ring-theoretical property. We say that P is a *Morita invariant* provided whenever a ring R satisfies P, $M_k(R)$ also satisfies P for all $k \geq 1$.

Remark

(a) Suppose that $\mathrm{st.r.}(R)$ equals 1 or 2. Then for any $k \geq 1$ we get that $\mathrm{st.r.}(M_k(R))$ also equals 1 or 2 respectively. In other words, stable range 1 and 2 properties are Morita invariants.

(b) Due to the increasing numerator in the formula of $\mathrm{st.r.}(M_k(R))$ we obtain that starting from some sufficiently large k all $M_k(R)$ are rings of stable range 2 whenever $1 < \mathrm{st.r.}(R) < \infty$ (in fact starting with size $k = \mathrm{st.r.}(R) - 1$).

In the above we have seen that stable range plays a role of linear independence of vectors in vector space but in case of ring elements. The other problem that is strictly connected to the linear independence is the existence of basis of free module and the number of its elements.

Definition

A ring R has *invariant basis number* (IBN) if for all $n \in \mathbb{N}$ and $p \geq 0$ the R-module isomorphism $R^{n+p} \cong R^n$ implies that $p = 0$, i.e. the number of generators of free R-module R^n is uniquely defined for any $n \geq 1$.

Remark

It can be proved that R has IBN if and only if for any pair of matrices $A \in M_{n,m}(R)$, $B \in M_{m,n}(R)$ such that $AB = I_n$, $BA = I_m$, one can infer that $n = m$. This reveals the left-right symmetry of the IBN notion.

Definition

A ring R is called *Dedekind-finite* if for any $a, b \in R$

$$ab = 1 \Longrightarrow ba = 1.$$

If for any $n \geq 1$ the ring $M_n(R)$ is Dedekind-finite then R is said to be *stably finite*.

Example

(1) Any commutative ring, one-sided Noetherian ring, reversible ring (i.e. $ab = 0$ implies $ba = 0$), Algebraic algebra over a field is Dedekind-finite ring. In particular, any finite ring or, more generally, a ring with finite index of nilpotency is a Dedekind-finite. Also, the direct product of Dedekind-finite rings is Dedekind-finite.

(2) All commutative rings and one-sided Noetherian rings are stably finite.

(3) For any group G the group ring $\mathbb{Z}G$ has IBN.

Remark

(1) IBN property and stable finiteness are Morita invariants.

(2) Clearly any Dedekind-finite ring has IBN, and every stably finite ring is Dedekind-finite. This inclusions are strict.

Example

Let R be a k-algebra generated over the field k by elements $\{s, t, u, v, w, x, y, z\}$ such that

$$\begin{pmatrix} s & u \\ t & v \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then R is a Dedekind-finite domain, but $M_2(R)$ is not Dedekind-finite. Moreover, by the remark above, $M_2(R)$ has IBN.

Definition

A ring R is called *purely infinite* if $R \cong R^2$ as R-modules.

Of course purely infinite rings have no IBN and $R \cong R^n$ for every $n \geq 1$, moreover any homomorphism $f : R \to S$ from the purely infinite ring R makes S also purely infinite.

Example

For any ring R the ring $S = \mathrm{End}_R(R^\infty)$ is purely infinite, where R^∞ denotes an infinite direct product of R.

Theorem

Every finite stable range ring R has IBN.

Remark

Since rings $\mathrm{End}_R(R^\infty)$ and $C(R)$ don't have IBN we conclude that they have infinite stable range. Speaking more generally, for any purely infinite ring R :

$$\mathrm{st.r.}(R) = \infty.$$

Theorem

If R is a ring of stable range 1, then it is stably finite.

Stably free modules

The essential role in algebra is played by projective objects. They can be described as objects P such that any morphism from P to some quotient-object A/B lifts to a morphism from P to the whole A. In case of module categories such objects are called projective modules and can be equivalently described as direct summands of free modules.

Definition

An R-module P is called *projective* if there exists an R-module Q and a free R-module F such that

$$P \oplus Q = F.$$

The category of finitely generated projective (left or right) R-modules is denoted by $\mathrm{Proj}(R)$.

In the case of $P \in \mathrm{Proj}(R)$ we can also write that

$$P \oplus Q \cong R^n$$

for some $Q \in \mathrm{Proj}(R)$ and $n \geq 1$. Obviously, free modules are projective. If an R-module Q (the complement of P) can be chosen to be free we obtain a new notion.

Definition

An R-module P is called *stably free* (of rank $n - m$) if

$$P \oplus R^m \cong R^n$$

for some m and n.

Example

(1) The unimodular row $\sigma = (x, y, z)$ is not completable to invertible matrix over $\mathbb{R}[x, y, z]/\langle x^2 + y^2 + z^2 - 1\rangle$, but $\mathrm{Ker}\sigma \oplus R \cong R^3$. Hence $\mathrm{Ker}\sigma$ is a stably free module of rank 2 that is not free.

(12) Any stably free R-module is free in case when R is a division ring, ring of integers or $M_n(k)$, where k is a field.

Theorem (Bass cancelation theorem)

Over a ring R of stable range n any stably free module of rank $m \geq n$ is free.

Corollary

Let R be a ring of stable range 1. Then any finitely generated stably free R-module is free.

Theorem (Bass Inequality)

Let R be a commutative Noetherian ring of Krull dimension $K.\mathrm{dim}(R)$. Then

$$\mathrm{st.r.}(R) \leq K.\mathrm{dim}(R) + 1.$$

Warfield has proved the following theorem that makes the connection between the stable range of endomorphism ring of some module and the decompositions of this module (such module property is called the n-substitution property).

Theorem (Warfield substitution theorem)

For the ring of endomorphisms E of the right R-module A the following conditions are equivalent:

(i) $\mathrm{st.r.}(E) \leq n$;

(ii) *if* $M \in \mathrm{Mod} - R$ *can be decomposed as*

$$M \cong A^n \oplus B_1 \cong A \oplus B_2$$

for some $B_1, B_2 \leq M$ *then there exist* $C, D \leq M$ *such that*

$$M = C \oplus D \oplus B_1 = C \oplus B_2.$$

Theorem

Let P and Q be right R-modules. Then

$$\mathrm{st.r.}(\mathrm{End}_R(P) \leq n, \ \mathrm{st.r.}(\mathrm{End}_R(Q) \leq n \Longrightarrow \mathrm{st.r.}(\mathrm{End}_R(P \oplus Q) \leq n.$$

Stable range

Theorem

Let R be a direct product of rings R_i, $i \in I$. Then

$$\mathrm{st.r.}(R) = \sup_{i \in I}(\mathrm{st.r.}(R_i)).$$

Corollary

If R is semisimple then it is a ring of stable range 1.

Definition

A ring R is called *semilocal* (*local*) if $R/J(R)$ is a semisimple (division) ring.

Since we have proved that $\mathrm{st.r.}(R) = \mathrm{st.r.}(R/J(R))$, and the ring is semisimple if and only if it is artinian with trivial Jacobson radical, then we obtain the following

Corollary

The stable range of any semilocal, local or artinian ring equals to 1.

In particular, any ring with finite number of elements (or more generally with finite number of ideals) has stable range 1.

Example

Since any $\mathbb{Z}/n\mathbb{Z}$ is a finite ring, then its stable range equals to 1. If one considers a triple of coprime elements $a, b, c \in \mathbb{Z}$ then $\overline{a} = a + \mathbb{Z}$, $\overline{b} = b + c\mathbb{Z}$ are coprime in $\mathbb{Z}/c\mathbb{Z}$. Since $\mathrm{st.r.}\mathbb{Z}/c\mathbb{Z} = 1$ then there exists $\overline{x} \in \mathbb{Z}/c\mathbb{Z}$ such that

$$\overline{a} + \overline{x}\overline{b} \in U(\mathbb{Z}/c\mathbb{Z}) = \{\overline{t} | t\mathbb{Z} + c\mathbb{Z} = \mathbb{Z}\}.$$

Therefore, $(a + xb)\mathbb{Z} + c\mathbb{Z} = \mathbb{Z}$ and $\mathrm{st.r.}(\mathbb{Z}) \leq 2$. Finally, note that there is no integer x such that $3 + 5x \in U(\mathbb{Z}) = \{\pm 1\}$, so $\mathrm{st.r.}(\mathbb{Z}) > 1$.

Proposition

Let R be a ring of stable range 1. If $aR = bR$ then $a = bu$, where u is an unit element of R.

Theorem

Let R be a ring in which every left principal ideal is a left annihilator for some element in R. Then the following statements are equivalent:

(1) $\mathrm{st.r.}(R) = 1$;

(2) *if $aR = bR$ then there exist units $u, v \in R$ such that $au = b$ and $bv = a$.*

Proposition

Let I be an ideal of a ring R. Then the following statements are equivalent:

(1) $\mathrm{st.r.}(R) = 1$;

(2) $\mathrm{st.r.}(R/I) = 1$ *and* $\mathrm{st.r.}(R/r(t)) = 1$;

(3) $\mathrm{st.r.}(R/I) = 1$ *and* $\mathrm{st.r.}(R/l(tI)) = 1$,

where $r(t)$ *— right annihilator for* t, $l(t)$ *— left annihilator for* t.

Proposition

A ring R has stable range 1 if and only if

$$aR + bR + cR = R$$

implies that $aR + (b + cr)R = R$ *for some* $r \in R$

Proposition

Stable range R *equal to 2 if and only if* $aR + bR + cR = R$ *implies that* $au + bv + cw = 1$ *for some* $u, v, w \in R$ *such that* $Ru + Rv = R$.

Von Neumann regular rings

A *von Neumann regular ring* is a ring such that for every $a \in R$ there exists an $x \in R$ such that $a = axa$.

Example

Every field (and every skew field) is von Neumann regular as a ring, since for $a \neq 0$ we can take $x = a^{-1}$. An integral domain is von Neumann regular if and only if it is a field. Another example of a von Neumann regular ring is the ring K_n of $n \times n$-matrices with entries from some field K. If r is the rank of $A \in R_n$ then there exist invertible matrices U and V such that

$$A = U \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} V$$

(where I_r is the identity $r \times r$-matrix). If we set $X = V^{-1}U^{-1}$, then

$$AXA = U \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} V = U \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} V = A.$$

A *Boolean ring* R is a ring in which $a^2 = a$ for every $a \in R$. Every Boolean ring is von Neumann regular.

Theorem

The following statements are equivalent for a ring R:

1) *R is von Neumann regular;*
2) *every principal right (left) ideal is generated by an idempotent;*
3) *every finitely generated right (left) ideal is generated by an idempotent*

A ring is *unit-regular* provided that for each $a \in R$ there exists a unit (i.e. an invertible element) u in R such that $a = aua$.

Example

Any direct product of fields, as well as any skew field, is unit-regular.

Proposition

A von Neumann regular ring R has stable range 1 if and only if it is unit-regular.

Recall that a commutative von Neumann regular ring has stable range 1.

Exchange, clean and idempotent stable range one rings

Studying the problem of idempotents lifting and exchange rings Nicholson [5] has introduced the notion of a clean ring. In this section we are going to review some of its useful properties.

[5] Nicholson W. K., *Lifting idempotents and exchange rings* // Trans. Amer. Alath. Soc. – 1977. – 229. – P. 269-278.

Definition

An element a in a ring R is called *clean* if it can be written as the sum of a unit and an idempotent of the ring R. A ring R is said to be *clean* if every element of R is clean.

Proposition

Every homomorphic image of a clean ring is clean.

Proposition

A direct product of rings $\prod R_i$ *is clean if and only if each ring* R_i *is clean.*

The next result is due to Han and Nicholson [6].

Theorem

A full matrix ring $M_n(R)$ is clean if the underlying ring R is clean.

[6] Han J., Nicholson W. K. *Extensions of clean rings* // Comm. Algebra. – 2001. – 29(6). – P. 2589-2595.

Definition

For an ideal I of a ring R we say that *idempotents lift modulo* I if for each element $x \in R$ such that $x - x^2 \in I$ there exists an idempotent e of R with the property $e - x \in I$. A ring R is an *exchange ring* if its idempotents lift modulo every left (right) ideal I of R.

Proposition

Every clean ring is an exchange ring.

Proposition

Every homomorphic image of an exchange ring is an exchange ring.

Definition

A ring R is *semiregular* if $R/J(R)$ is a von Neumann regular ring and idempotents can be lifted modulo $J(R)$.

Proposition

Every semiregular ring is an exchange ring.

Proposition

A ring R with central idempotents is clean if and only if it is an exchange ring.

Definition

A ring R is said to be *reduced* if it has no (nonzero) nilpotent elements.

Proposition

A reduced ring is a ring with central idempotents.

Definition

A ring R is called a *potent* ring if its idempotents can be lifted modulo $J(R)$ and every left (right) ideal not contained in $J(R)$ contains a nonzero idempotent. A ring R is said to be *semipotent* if each left (right) ideal of R not contained in $J(R)$ contains a nonzero idempotent.

It is well known that potent and semipotent conditions on ring are left-right symmetric. Obviously, any potent ring is semipotent and every exchange ring is potent.

Example

Let D denote a division ring and let S be a subring of containing 1. Define

$$R = R(D, S) = \{(x_1, x_2, \ldots, x_n, s) \mid n \geq 1, x_i \in D, s \in S\}.$$

Then R is a ring (with componentwise operations) and R is an exchange ring if and only if the same true of S. In fact S is a homomorphic image of R while if S is an exchange ring, the same is true of R by a componentwise calculation. Furthermore every nonzero left (right) ideal of $R(D, S)$ contains a nonzero idempotent so $J(R) = 0$. Then $R(D, S)$ is a commutative potent ring which is not exchange ring.

Proposition

Every clean ring is semipotent.

Proposition

Every clean ring is potent.

Gathering all results, we obtain the following chain of ring classes:

$$\text{clean} \Rightarrow \text{exchange} \Rightarrow \text{potent} \Rightarrow \text{semipotent}.$$

This inclusions are known to be irreversibe.

In particular, Camillo and Yu [7] have proved that the ring in the well-known Bergman's [8] example is an exchange ring which is not clean. Meanwhile, Nicholson [9] has shown that a potent ring need not to be an exchange one and Nicholson and Zhou [10] have constructed an example of semipotent ring which is not a potent one.

[7] Camillo V. P., Yu H. P. *Exchange rings, units and idempotents* // Comm. Algebra. – 1994. – 22(12). – P. 4737-4749.

[8] Handelman D. *Perspectivity and cancellatiorl in regular rings* // J. Algebra. – 1977. – 48. – P. 1-16.

[9] Nicholson W. K., *Lifting idempotents and exchange rings* // Trans. Amer. Alath. Soc. – 1977. – 229. – P. 269-278.

[10] Nicholson W. K., Zhou Y. *Clean general rings* //J. Algebra. – 2005. – 291(1). – P. 297-311.

Definition

A ring R is said to be a ring of an *idempotent stable range 1* if for any $a, b \in R$ such that $Ra + Rb = R$ there exists an idempotent $e \in R$ such that $a + eb$ is a unit of R.

Proposition

Every ring of an idempotent stable range 1 is clean.

Proposition

Let R be a ring with central idempotent. Then the following statements are equivalent:

(1) *R is a ring of an idempotent stable range 1;*

(2) *R is a clean ring;*

(3) *is an exchange ring.*

Corollary

Any division ring, boolean ring and local ring is clean.

As an immediate corollary, we have the following result.

Proposition

A domain is clean if and only if it is a local domain.

Therefore every semilocal domain which is not local is an example of a ring of stable range 1 which is not a ring of idempotent stable range 1.

Definition

A left R-module M is said to have the *exchange property* if for any module X a decomposition

$$X = M' \oplus Y = \bigoplus_{i \in I} N_i$$

where $M' \cong M$, there exist submodules $N_i{}' \subseteq N_i$ for each $i \in I$ such that

$$X = M' \oplus (\bigoplus_{i \in I} N_i').$$

If this condition holds for any finite set I (equivalently for $|I| = 2$) the module M is said to have the *finite exchange property*.

Theorem

Let R be a ring. The following conditions are equivalent for a left R-module M:

(1) $\mathrm{End}_R(M)$ *is an exchange ring;*

(2) M *has the finite exchange property.*

Definition

A commutative ring R is called a *Gelfand ring* if whenever $a + b = 1$ there exist $r, s \in R$ such that

$$(1 + ar)(1 + bs) = 0.$$

Definition

A commutative ring is called a *PM-ring* if every prime ideal is contained in a unique maximal ideal.

It is known that for any topological space X the ring $C(X)$ consisting of all real-valued continuous functions on X under the pointwise operations is always a PM-ring [11]. Other obvious examples of PM-rings include commutative von Neumann regular rings, local rings and zero-dimensional rings.

[11] Gillman L., Henriksen M. *Rings of continuous function in which every finitely generated ideal is principal* // Trans. Amer. Math. Soc. – 1956. – 82(2). – P. 366-391.

Theorem

A commutative ring R is a PM-ring if and only if R is a Gelfand ring.

We obtain the following corollaries of Theorem:

(1) A commutative von Neumann regular ring R is a Gelfand ring; since the equality $a + b = 1$ implies that $(1 - ax)a = (1 - ax)(1 - b) = 0$, where $axa = a$ for some $x \in R$.

(2) Let R be a zero-dimensional ring. If N is the nilradical of R, then R/N is obviously a von Neumann regular ring and therefore the equality

$$\overline{a}(\overline{1} - \overline{ax}) = \overline{0}$$

lifts to $a(1 - ax) \in N$ in R or $a^n(1 - ax)^n = 0$ for some $n \in \mathbb{N}$.

(3) In a local ring R where $a + b = 1$ implies that at least a or b is a unit and thus we have the equality

$$(1 + ar)(1 + bs) = 0$$

with $r = -a^{-1}$ or $s = -b^{-1}$, so a local ring is a Gelfand ring.

Proposition

A commutative clean ring is a PM-ring.

Morphic rings

A well-known theorem of Erlich [12] states that a map ϕ in an endomorphism ring of a R-module M is unit-regular if and only if it is regular and

$$M/\mathrm{Im}(\phi) \cong \mathrm{Ker}(\phi).$$

[12] Erlich G. *Units and one-sided units in regular rings* // Trans. Amer. Math. Soc. – 1976. – 216. – P. 81-90.

Definition

An element a of a ring R is called *left (right) morphic* if $R/Ra \cong l(a)$ (respectively $R/aR \cong r(a)$). The ring R is called *a left (right) morphic ring* if every its element is left (right) morphic.

$l(a)$ — left annihilator of a, $r(a)$ — right annihilator of a.

Definition

An element $a \in R$ is called a *von Neumann (unit-) regular* if $axa = a$ for some (unit) $x \in R$.

If a is a unit-regular element then $aua = a$, where u is a unit. If we take $e = ua$ then $a = u^{-1}e$ is left morphic, because $e^2 = e$ and every idempotent element is a left morphic element.

Proposition

Every unit-regular ring is left and right morphic.

Example

The converse to the latter statement is false: $\mathbb{Z}_4$ is left morphic but it is not unit-regular.

Example

A polynomial ring $R[x]$ is never left or right morphic, because $l(x) = 0$ and $x \notin U(R[x])$, and the only left (right) morphic domains are the division rings.

The following result gives another source of examples of left morphic rings.

Proposition

If a ring R has a unique left ideal $I \neq 0$ then R is left morphic.

A commutative left morphic ring will be simply called *a morphic ring*.

Theorem

Let R be a commutative morphic ring. Then the following properties hold:

(1) *finite intersections of principal ideals of R are principal;*

(2) *R is a Bezout ring.*

The End