Conditions for stable range of an elementary divisor rings

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Hermitian and elementary divisor rings

Elementary divisor rings were defined by I. Kaplansky [1]. Any elementary divisor ring is an Hermitian ring and Hermitian ring is a Bezout ring [2]. Gillman and Henriksen [3] constructed an example of a commutative Bezout ring which is not an Hermitian ring and an example a commutative Hermitian ring which is not an elementary divisor ring.

[1] Kaplansky I. *Elementary divisors and modules* // Trans. Amer. Math. Soc. – 1949. – P. 464-491.

[2] Gillman L., Henriksen M. *Some remarks about elementary divisor rings* // Trans. Amer. Math. Soc. – 1956. – 82. – P. 362-365.

[3] Gillman L., Henriksen M. *Rings of continuous function in which every finitely generated ideal is principal* // Trans. Amer. Math. Soc. – 1956. – 82(2). – P. 366-391.

We have the following chain of rings classes

$$\text{Bezout ring} \subset \text{Hermitian ring} \subset \text{Elementary divisor ring}$$

This inclusions are irreversible.

Definition

It is said that matrices A and B over R are *equivalent* ($A \sim B$) if there are invertible matrices P and Q over R of appropriate sizes such that $A = PBQ$. Say that a matrix A over a ring R *admits a canonical diagonal reduction* if it is equivalent to a diagonal matrix

$$\begin{pmatrix} \varepsilon_1 & 0 & \dots & 0 & 0 & \dots 0 \\ 0 & \varepsilon_2 & \dots & 0 & 0 & \dots 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ 0 & 0 & \dots & \varepsilon_r & 0 & \dots 0 \\ 0 & 0 & \dots & 0 & 0 & \dots 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & \dots 0 \end{pmatrix},$$

where $R\varepsilon_{i+1}R \subseteq \varepsilon_i R \cap R\varepsilon_i$ for any $i \in \{1, \dots, r-1\}$. If every matrix over R admits a canonical diagonal reduction then R is said to be an *elementary divisor ring*.

Definition

If every 1×2 (2×1) matrix over R admits a canonical diagonal reduction then, R is called *a right (left) Hermitian ring* . It is clear that in the case of commutative rings every right Hermitian ring is a left Hermitian ring. *An Hermitian ring* is a ring which is a both right and left Hermitian ring.

Remark

Let R be a right Hermitian ring. Then for any $a, b \in R$ there exists an invertible 2×2-matrix P and there exists $d \in R$ such that

$$(a, b)P = (d, 0).$$

Suppose that

$$P = \begin{pmatrix} x & u \\ y & v \end{pmatrix} \quad \text{and} \quad P^{-1} = \begin{pmatrix} a_1 & b_1 \\ r & s \end{pmatrix}.$$

Then $ax + by = d$, $a = da_1$, $b = db_1$, $a_1R + b_1R = R$ and $aR + bR = dR$, i.e. R is a right Bezout ring.

Theorem (Henriksen's criterion)

A commutative Bezout ring R is an Hermitian ring if and only if for $a, b \in R$ there exist $d, a_1, b_1 \in R$ such that $a = a_1 d$, $b = b_1 d$, $a_1 R + b_1 R = R$. [a]

[a] Gillman L., Henriksen M. *Some remarks about elementary divisor rings* // Trans. Amer. Math. Soc. – 1956. – 82. – P. 362-365.

Theorem

A commutative Bezout ring R is an Hermitian ring if and only if $\mathrm{st.r.}(R) = 2$.

Recall a ring R is a ring of stable range 2 if $aR + bR + cR = R$ we have $(a + cx)R + (b + cy)R = R$ for some $x, y \in R$.

Theorem

Let R be a commutative ring. If all 1×2, 2×1 and 2×2-matrices over R admit a canonical diagonal reduction then R is an elementary divisor ring. [[a]]

[a] Kaplansky I. *Elementary divisors and modules* // Trans. Amer. Math. Soc. – 1949. – P. 464-491.

Theorem (Kaplansky's criterion)

A commutative Hermitian ring is an elementary divisor ring if and only if $aR + bR + cR = R$ *and there exist elements* p *and* q *such that* $paR + (pb + qc)R = R$. [a],[b]

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[a] Kaplansky I. *Elementary divisors and modules* // Trans. Amer. Math. Soc. – 1949. – P. 464-491.

[b] Gillman L., Henriksen M. *Some remarks about elementary divisor rings* // Trans. Amer. Math. Soc. – 1956. – 82. – P. 362-365.

Adequate rings

We consider commutative rings, in which any nonzero prime ideal is contained in a unique maximal ideal.

Definition

An element a of a commutative ring R is called *adequate to an element b* if there exist elements $r, s \in R$ such that

1. $a = rs$;
2. $rR + bR = R$;
3. $s'R + bR \neq R$ for any nonunit divisor s' of s.

Definition

An element a in R is called *adequate* if it is adequate to every element b of R. A commutative ring is called an *adequate ring* if all its nonzero elements are adequate. If even a zero element of an adequate ring is adequate such a ring is called *everywhere adequate*.

Although any everywhere adequate ring is an adequate ring the converse is not always true. For example, a ring of integers $\mathbb{Z}$ is adequate but not everywhere adequate.

Since any commutative principal ideal domain is a factorial domain [4], we have the following result.

Theorem

A commutative principal ideal domain is an adequate domain.

[4] Kaplansky I. *Commutative rings* // The University of Chicago Press, Chicago and London. – 1974. – 180 p.

A ring of entire functions on the complex plane is a commutative adequate Bezout domain. Obviously, R is not a principal ideal domain.

Theorem

Every commutative von Neumann regular ring is an everywhere adequate ring.

Recall that R is a von Neumann regular ring of for any $a \in R$ we have $axa = a$ for some $x \in R$.

Theorem

Every nonzero prime ideal of an adequate Bezout ring R is contained in a unique maximal ideal of R.

Example (Henriksen's example)

Consider the subring

$$R = \{z_0 + a_1 x + a_2 x^2 + \dots | z_0 \in \mathbb{Z}, a_i \in \mathbb{Q}\}$$

of the ring of formal power series over $\mathbb{Q}$. [a]

[a] Henriksen M. *Some remarks on elementary divisor rings* // Michigan Math. J. – 1955-1956. – 3. – P. 159-163.

There is example Bezout domain (= elementary divisor domain) which is not adequate ring.

Stable range of adequate rings

We will calculate of stable range of an adequate Bezout ring.

Theorem

If R is an adequate Bezout ring then $\mathrm{st.r.}(R) = 2$.

Theorem

An adequate Bezout ring is a Hermitian ring.

Theorem

If R is an adequate Bezout ring and $J(R) \neq 0$ then $\mathrm{st.r.}(R) = 1$.

Theorem

Adequate ring is an elementary divisor rings if and only if it is Bezout ring.

Theorem

A commutative Bezout domain is an adequate domain if and only if R/aR is a semi-regular ring for any nonzero element $a \in R$.

Recall that a ring R is a ring of stable range 1 if $aR + bR = R$ we have $(a + bx)R = R$ for some $x \in R$.

Zero-adequate and everywhere adequate rings

Now we are going to study commutative rings for which zero is an adequate element. The structure of these allows us to construct more examples of adequate rings.

Theorem

Let a be an adequate element of a commutative Bezout ring. Then zero is an adequate element of the factor-ring R/aR.

Theorem

Let 0 be an adequate element of a commutative Bezout ring R. Then R is a ring of idempotent stable range 1.

Theorem

Let R be a commutative Bezout ring and 0 be an adequate element of R. Then R is a clean ring.

Recall that a ring R is a clean ring if for any $a \in R$ we have $a = u + e$, where u is invertible and $e^2 = 2$.

Now, we turn to answer another question. Let R be a commutative Bezout ring and zero be an adequate element in the factor-ring R/aR. Is the element a adequate in R? The answer is affirmative in the case of a commutative Bezout domain, and it is given by the following theorem.

Theorem

Let R be a commutative Bezout domain. If zero is an adequate element of the factor-ring R/aR then a is an adequate element of the domain R.

We will consider everywhere adequate rings. The main examples of everywhere adequate rings include von Neumann regular rings and valuation rings.

Theorem

Any everywhere adequate Bezout ring is a clean ring.

Theorem

Any everywhere adequate Bezout ring is a PM-ring.

Recall that R is a PM-ring if for any prime ideal contained in unique maximal ideal.

Theorem

Any everywhere adequate Bezout ring is a ring of idempotent stable range 1.

Recall that R is a ring of idempotent stable range 1 if $aR + bR = R$ we have $a + be$ — invertible element, where $e^2 = 2 \in R$.

Neat range 1

As proved a commutative Hermitian ring is precisely a Bezout ring of stable range 2. There is a similar description of commutative elementary divisor rings.

Definition

A commutative ring R is said to be a *ring of neat range 1* if for any $a, b \in R$ such that $aR + bR = R$ and for any $c \in R \setminus \{0\}$ there exists $u, v, t \in R$ such that $a + bt = uv$, where $uR + cR = R$, $vR + (1 - c)R = R$, and $uR + vR = R$.

This denote, that $R/(a + bt)R$ is a clean ring.

An obvious example of a ring of neat range 1 is any ring of stable range 1.

Proposition

Let R be a commutative Bezout ring and let a be an element of R such that for any $c \in R$ there exist $u, v \in R$ such that $a = uv$ where $uR + cR = R$, $vR + (1 - c)R = R$, and $uR + vR = R$. Then R/aR is a clean ring.

Theorem

A commutative Hermitian ring R is an elementary divisor ring if and only if R is a ring of neat range 1.

A new example of a ring of neat range 1 is a Dirichlet domain.

Definition

A commutative domain R is said to be a *Dirichlet domain* if for any elements $a, b \in R$ such that $aR + bR = R$ there exist an element $t \in R$ such that the element $a + bt$ is an atom of R.

Recall that element a is an atom of R if decomposition $a = bc$ we have that b or c is invertible element of R.

The ring of integers $\mathbb{Z}$ is an obvious example of a Dirichlet ring (by Dirichlet Theorem). A ring $\mathbb{C}[x]$ is not a Dirichlet ring. Really, $x^2 \cdot \mathbb{C}[x] + (x^2+1) \cdot \mathbb{C}[x] = \mathbb{C}[x]$ then $\deg(x^2 f(x) + (x^2+1)g(x)) \neq 1$ for any $f(x)$ and $g(x)$ in $\mathbb{C}[x]$.

Let R be a commutative Bezout domain and p be an atom of R. Then $pR \in \mathrm{mspec} R$ and R/pR is a field.

Theorem

A Dirichlet Bezout domain is an elementary divisor ring.

Morphic rings

Proposition

If R is a commutative Bezout domain and $0 \neq a \in R$, then R/aR is a morphic ring.

Theorem

Let R be a commutative Bezout domain. Then for any nonzero element $a \in R$, R/aR is a morphic ring.

Recall that R is a morphic ring if for any $a \in R$ there exist $b \in R$ such that $bR = Ann(a)$.

As a consequence of this fact we can give an example of a commutative morphic ring that is not clean. It shows that an answer to a question put by Nicholson in [5] is negative.

Example

Let R be a Henriksen's example:

$$R = \{z_0 + a_1 x + a_1 x^2 + \dots \mid z_0 \in \mathbb{Z}, a_i \in \mathbb{Q}, i = 1, 2, \dots\}.$$

We have shown that R is a commutative Bezout domain [a]. The factor ring R/xR is a morphic ring but it is not clean since a homomorphic image of the ideal $N = \{a_1 x + a_1 x^2 + \dots | a_i \in \mathbb{Q}, i = 1, 2, \dots\}$ is an ideal N/xR that is prime, but belongs to all maximal ideals in the factor ring R/xR. That is why R/xR is not clean, because any clean ring has to be a PM-ring. Note that $xR \neq N$, since $x/2 \in N$ but $x/2 \notin xR$.

[a] Henriksen M. *Some remarks on elementary divisor rings* // Michigan Math. J. – 1955-1956. – 3. – P. 159-163.

[5] Nicholson W. K., Sanchez Campos E., *Rings with the dual of the isomorphism theorem* // J.Algebra. – 2004. – 271. – P. 391-406.

In his pioneering paper Kaplansky [6] raised the following question: if $aR = bR$ in a ring R then are a and b necessarily right associates? Developing these ideas Canfell [7] introduced the concept of uniquely generated set of principal ideals.

Definition

Let $\{a_i R | i = 1, 2, \ldots, n\}$ be a finite set of principal ideals of a commutative ring R. It is said that this set of principal ideals is *uniquely generated* if whenever $a_1 R = b_1 R, \ldots, a_n R = b_n R$ there exist elements $u_1, \ldots, u_n \in R$ such that $a_i = b_i u_i$, $i = 1, 2, \ldots, n$, and $u_1 R + u_2 R + \ldots + u_n R = R$. *The dimension of a commutative ring R* (denoted by $\dim(R)$) is the least integer n such that every set of $n+1$ principal ideals is uniquely generated.

[6] Kaplansky I. *Elementary divisors and modules* // Trans. Amer. Math. Soc. – 1949. – P. 464-491.

[7] Canfell M. J. *Uniqueness of generators of principal ideals in rings of continuous function* // Proc. Amer. Math. Soc. – 1970. – 26(4). – P. 517-573.

Canfell obtained the characterizations of n-dimensional F-spaces in terms of rings of continuous real-valued and complex-valued functions defined on such spaces. Extending the uniqueness notion of principal ideals generators he gave an algebraic characterization of the concept "n-dimensional".

We will show that in the case of a commutative morphic ring the property $\dim(R) = 1$ is equivalent to the stable range 2 condition.

Theorem

If R is a commutative Bezout ring with $\dim(R) = 1$, then $\mathrm{st.r.}(R) = 2$.

Theorem

Let R be a commutative morphic ring of stable range 2. Then $\dim(R) = 1$.

Theorem

A commutative morphic ring R is the ring of stable range 2 if and only if $\dim(R) = 1$.

Theorem

If R is an elementary divisor domain and $a \in R \setminus \{0\}$, then the factor ring R/aR is a morphic ring of neat range 1.

Gelfand range 1 and Bezout PM^*-domains

Definition

A nonzero element a of a commutative ring is called *a PM-element* if the factor-ring R/aR is a PM-ring.

Proposition

For a commutative ring R the following statements are equivalent:

(1) *a is a PM-element;*

(2) *for each prime ideal P containing the element $a \in P$ there exists a unique maximal ideal such that $P \subset M$.*

We obtain the following results.

Proposition

A commutative domain R is a domain in which any nonzero prime ideal is contained in a unique maximal ideal if and only if any nonzero element of R is a PM-element.

Proposition

An element a of a commutative Bezout domain R is a PM-element if and only if for every elements $b, c \in R$ such that $aR + bR + cR = R$ an element a can be represented as $a = rs$, where $rR + bR = R$ and $sR + cR = R$.

Theorem

A commutative Bezout domain in which any nonzero prime ideal is contained in a unique maximal ideal is an elementary divisor ring.

In the case of a commutative Bezout domain this result can be clarified and improved.

Definition

Let R be a commutative Bezout domain. A ring R is called a ring of *Gelfand range 1* if for any $a, b \in R$ such that $aR + bR = R$ there exists an element $t \in R$ such that $a + bt$ is a PM-element.

Since every unit is a *PM*-element we have the following result.

Proposition

Any commutative Bezout domain of stable range 1 is a ring of Gelfand range 1.

Theorem

A commutative Bezout domain is an elementary divisor ring if and only if it is a ring of Gelfand range 1.

Theorem

A commutative ring of stable range 1 is an elementary divisor ring if and only if it is a Bezout ring.

Hence the open problem "is every commutative Bezout domain which is not a ring of stable range 1 an elementary divisor ring" is equivalent to the problem does every commutative Bezout domain R which is not a ring of stable range 1 contains a non-unit element $a \in R$ such that $st.r.(R/aR) = 1$.

Full matrices over elementary divisor rings

Definition

A matrix $A \in R_n$ is called *full* if $R_n A R_n = R_n$. We denote by $F(R_n)$ the class of all full matrices in R_n, where $R_n = M_n(R)$.

Theorem

Let R be a commutative Bezout ring of stable range 2. Let $A, B \in F(R_2)$ be such that $AR_2 + BR_2 = R_2$. Suppose that B admits a canonical diagonal reduction. Then there exists a full matrix $T \in F(R_2)$ such that $A + BT$ is an invertible matrix.

Theorem

Let R be a commutative elementary divisor ring. If $A, B \in F(R_2)$ and $AR_2 + BR_2 = R_2$, then there exists a full matrix $T \in F(R_2)$ such that $A + BT$ is an invertible matrix.

Moreover, we derive the following result.

Theorem

Let R be a commutative elementary divisor ring. Then for any full matrices $A, B \in F(R_2)$ there exist full matrices $Q_1, Q_2, P \in R_2$ such that $A = BQ_1 + P$, $B = PQ_2$.

Dyadic range 1

We describe commutative elementary divisor rings based on the concept of a ring of dyadic range 1.
Let R be a associative ring with unit and $1 \neq 0$.

Definition

Let $a, b \in R$ and $aR + bR = R$. Will say that a pair (a, b) *has a right diadem*, or (a, b) is *a right dyadic pair*, if there exists an element $\lambda \in R$ such that for the element $a + b\lambda$ and any elements $c, d \in R$ such that $(a + b\lambda)R + cR + dR = R$ there exists an element $\mu \in R$ such that $(a + b\lambda)R + (c + d\mu)R = R$. Call an element $a + b\lambda$ *a right diadem* of the pair (a, b). A left diadem and a left diadic pair can be introduced analogously. A right and left diadem we will simply call a *diadem*.

Example

An obvious example of a diadic pair (that may also be called a *trivial diadic pair*) is (a, u), where u is an invertible element of a ring R and a is any element of R. Here, $u + a0$ and $a + (-au^{-1} + 1)u$ are right diadems of the pair (a, u).
To obtain a nontrivial example, take a pair $(a, a + u)$, where $a \in R$ and u is an invertible element of R. Therefore, $a + (a + u) - 1$, $(a + u) + a(-1)$ are a right diadems of the pair $(a, a + u)$.

Definition

We say that a ring R is a ring of *a right dyadic range 1* if for any elements a, b the equality $aR + bR = R$ implies that a pair (a, b) has a right diadem. Similarly, we define a ring of a left dyadic range 1. A ring of right and left dyadic range 1 is called *a ring of dyadic range 1*.

Example

Any ring of stable range 1 is a ring of dyadic range 1.

Moreover, we have the following result.

Theorem

A Bezout ring of right dyadic range 1 is a ring of stable range 2.

Proposition

A ring R is of right dyadic range 1 if and only if $R/J(R)$ is a ring of right dyadic range 1.

We will use the following result.

Proposition

Let R be a commutative ring and (a, b) is a dyadic pair. An element $a + b\lambda$ is a diadem if and only if the factor-ring $R/(a + b\lambda)R$ is a ring of stable range 1.

Proposition

Let R be a commutative Bezout ring of dyadic range 1. Then for any divisor α of the diadem $a + b\lambda$ and any elements $c, d \in R$ such that $\alpha R + cR + dR = R$ there exists an element $\mu \in R$ such that $\alpha R + (c + d\mu)R = R$.

Theorem

A commutative Bezout ring is an elementary divisor ring if and and only if it is a ring of dyadic range 1.

As a consequence of this theorem we have the following.

Proposition

Let R be a commutative Bezout ring of dyadic range 1. Then for any ideal I of R the factor-ring R/I is a ring of dyadic range 1.

Proposition

Let R is a commutative semihereditary Bezout ring. If for any regular element (non zero divisor) $r \in R$ the factor-ring R/rR is a ring of dyadic range 1 then R is a ring of dyadic range 1.

Consequently, we have that any example of a commutative Hermitian ring which is not an elementary divisor ring [8] is an example of a commutative Bezout ring of stable range 2 which is not of a ring of dyadic range 1.

[8] Gillman L., Henriksen M. *Rings of continuous function in which every finitely generated ideal is principal* // Trans. Amer. Math. Soc. –1956. – 82(2). –P. 366-391.

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